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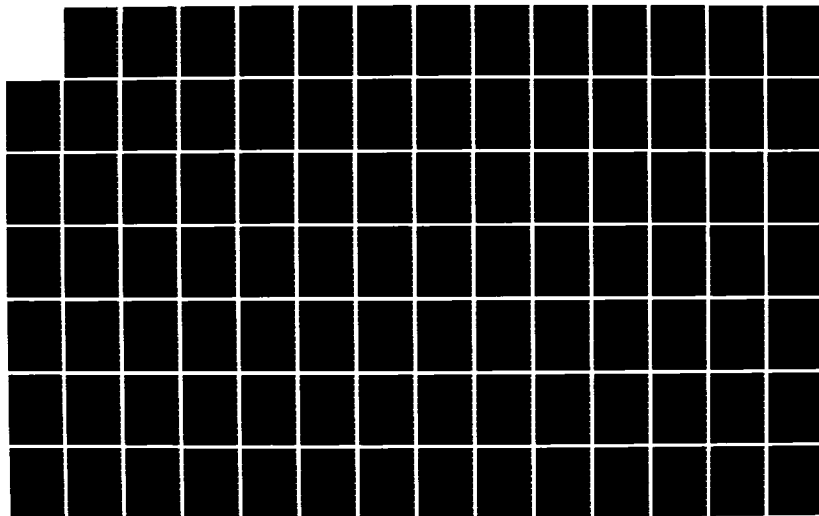
WAVE PROPAGATION AND DYNAMICS OF LATTICE STRUCTURES(U)  
NEA CAMBRIDGE MA J H WILLIAMS ET AL. 01 MAY 84  
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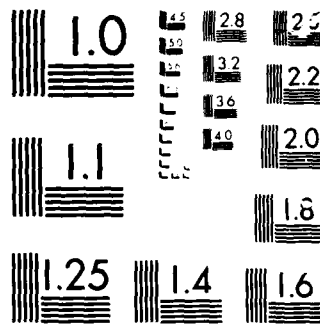
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<p>The application of the transfer matrix method to the analysis of wave propagation and vibration in periodic structures is introduced. Analyses of a one-dimensional rod, a three-dimensional planar lattice structure and a three-dimensional tetrahedral truss are given to illustrate the general approach in applying the transfer matrix method. In addition, a numerical example is given.</p> <p>The frequency response functions for specific locations in a one-dimensional rod due to an excitation at a particular location are obtained using a basic language computer program. The responses at one location in the rod due to an impulse excitation, a square pulse excitation and a triangular pulse excitation at a second location are also obtained.</p>			
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## INTRODUCTION

A periodic structure is one which consists of a number of identical substructures, called periodic units, coupled together to form the entire structure. There are in general two types of periodic structures - simple periodic structures and compound periodic structures. A simple periodic structure is one which consists of basic periodic units which cannot be divided further into identical subunits. Figure 1 shows an example of such a structure. Figure 1 shows a uniform beam supported at spacing  $\ell$ . Oscillators are attached at midspan between the supports as shown. The oscillators are modeled as consisting of mass ( $m_d$ ), elastic stiffness (spring constant  $k_d$ ) and damping (dashpot constant  $c_d$ ). A compound periodic structure consists of periodic units which in themselves are composed of identical periodic subunits. Thus a periodic unit in a compound periodic structure is itself another periodic structure. Figure 2 shows an example of a compound periodic structure. Figure 2 shows a uniform beam simply supported at spacing  $\ell$  where oscillators are equally spaced within the spans as shown.

Periodic structures have been analyzed using the concepts of a "propagation constant" [1-3] and also by the "transfer matrix method" [4]. The concept of propagation constants as applied to wave propagation in periodic structures is reviewed in Appendix A. However, the derivation of propagation constants is very cumbersome, requiring the solution of fourth-order partial differential equations in most cases. On the other hand, due to recent advances in transfer matrices [5,6], the transfer matrix method appears to be less cumbersome. This approach permits a simple treatment of periodic units with complicated configurations and furthermore, a matrix formulation is most suitable for periodic structures of finite total length since the imposition of boundary conditions at the ends of the structure is straight forward [7].

In this report, an attempt is made to utilize the transfer matrix method for analyzing wave propagation in periodic structures. Three examples are given to illustrate the general approach. The case of longitudinal vibration and wave propagation in a rod is considered first. This

is followed by more complicated problems which include vibration and wave propagation in both a two-dimensional 3-bay planar lattice structure and a three-dimensional tetrahedral truss.

## ANALYSIS

### THE TRANSFER MATRIX METHOD

#### State Vectors

The identification of a state vector is important in applying the transfer matrix method to wave propagation and vibration of structures. The state vector  $z$  at a point  $i$  of an elastic system is a column vector, the components of which are the displacements at the point and the corresponding internal forces.

For example, in the longitudinal vibration of a straight rod, the state vector  $z$  consists of components  $u$  and  $N$ , where  $u$  is the longitudinal displacement and  $N$  is the axial force. As another example, for the analysis of the flexural vibration of a Timoshenko beam, the state vector consists of components  $w$ ,  $\psi$ ,  $M$  and  $V$  where  $w$  is the transverse displacement,  $\psi$  is the rotation in radians of the cross-section, and  $M$  and  $V$  are the moment and the shear force, respectively.

In the analysis of a structure, if the time histories of the state vectors for specified locations in the structure are known, the vibration characteristics, as well as the wave propagation characteristics, can be determined.

#### Transfer Matrices

A transfer matrix relates the state vector at a point in a structure to the state vector at another point in the same structure. It is an  $n \times n$  matrix, where  $n$  refers to the number of components in the corresponding state vector.

Some common transfer matrices are derived in Appendices B through E. The transfer matrices (from left to right and from right to left, the significance of which is explained later in this

section) for longitudinal vibration in an elastic rod with distributed mass are derived in Appendix B. The transfer matrix for longitudinal vibration in an elastic rod with distributed mass and damping is derived in Appendix C. The transfer matrix for flexural vibration in an elastic beam with distributed mass, including the effects of shear deflection and rotary inertia is derived in Appendix D and the transfer matrix for both longitudinal and flexural vibrations in a bar with distributed mass, including the effects of shear deflection and rotary inertia, is derived in Appendix E.

For example, the transfer matrix for longitudinal vibration in a straight rod relates the state vectors at the two end points as (refer to Appendix B)

$$\begin{Bmatrix} u \\ V \end{Bmatrix}_R = \begin{bmatrix} \cos \theta & \frac{\ell \sin \theta}{EA \theta} \\ -\mu \ell \omega^2 \frac{\sin \theta}{\theta} & \cos \theta \end{bmatrix} \begin{Bmatrix} u \\ V \end{Bmatrix}_L \quad (1)$$

and for flexural vibration in a Timoshenko beam, the transfer matrix relates the state vectors at the two end points as (refer to Appendix D)

$$\begin{Bmatrix} -w \\ \psi \\ M \\ V \end{Bmatrix}_R = \begin{bmatrix} c_0 - \alpha_2 & \ell[c_1 - (\sigma + \tau)c_3] \\ \frac{\beta^4}{\ell} c_3 & c_0 - \tau c_2 \\ \frac{\beta^4 EJ}{\ell^2} c_2 & \frac{EJ}{\ell} [-\tau c_1 + (\beta^4 + \tau^2)c_3] \\ \frac{\beta^4 EJ}{\ell^3} (c_1 - \alpha_3) & \frac{\beta^4 EJ}{\ell^2} c_2 \end{bmatrix} \begin{bmatrix} \frac{\ell^2 c_2}{EJ} & \frac{\ell^3}{\beta^4 EJ} [-\alpha_1 + (\beta^4 + \sigma^2)c_3] \\ \frac{\ell^1}{EJ} (c_1 - \tau c_3) & \frac{\ell^2}{EJ} c_2 \\ c_0 - \tau c_2 & \ell[c_1 - (\sigma + \tau)c_3] \\ \frac{\beta^4}{\ell} c_3 & c_0 - \alpha_2 \end{bmatrix} \begin{Bmatrix} -w \\ \psi \\ M \\ V \end{Bmatrix}_L \quad (2)$$

where the subscripts  $R$  and  $L$  denote the right and left end state variables faces, respectively. For

the definitions of the variables used in eqns. (1) and (2), refer to Appendices B and D respectively.

Appendix C also shows the general approach for deriving the transfer matrix for longitudinal vibration in an elastic rod with damping. However, since the introduction of damping complicates the transfer matrices considerably, the problem of damping will not be considered here.

#### Uses of Transfer Matrices

As mentioned previously, a transfer matrix is used to relate the state vectors at two specified points in a structure. Notice that due to the sign conventions chosen in deriving transfer matrices, a transfer matrix becomes different when a left end state vector of a particular element is written in terms of the right end state vector as opposed to when the right end state vector is written in terms of the left end state vector.

For example, for longitudinal vibration in a straight rod (refer to Appendix B), the transfer matrix from left to right relates the state vectors at the two ends of the rod as

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_R = \begin{bmatrix} \cos \theta & \frac{\ell}{EA} \frac{\sin \theta}{\theta} \\ -\mu \ell \omega^2 \frac{\sin \theta}{\theta} & \cos \theta \end{bmatrix} \begin{Bmatrix} u \\ N \end{Bmatrix}_L \quad (3)$$

and the transfer matrix from right to left relates the state vectors at the two ends as

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_L = \begin{bmatrix} \cos \theta & -\frac{\ell}{EA} \frac{\sin \theta}{\theta} \\ \mu \ell \omega^2 \frac{\sin \theta}{\theta} & \cos \theta \end{bmatrix} \begin{Bmatrix} u \\ N \end{Bmatrix}_R \quad (4)$$

It can be shown that the transfer matrix in eqn. (4) can also be obtained by simply taking the inverse of the transfer matrix in eqn. (3) and vice versa (refer to eqn. (B16)).

One of the useful and interesting characteristics of the transfer matrix method is that transfer matrices can be multiplied to form another transfer matrix which represents a larger section of a structure. As a simple example, consider the rod of length  $2\ell$  as shown in Figure 3. The rod is



assumed to consist of two identical elements, each of length  $\ell$ . Using eqn. (3),

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_B = \begin{bmatrix} \cos \theta & \frac{\ell}{EA} \frac{\sin \theta}{\theta} \\ -\mu \ell \omega^2 \frac{\sin \theta}{\theta} & \cos \theta \end{bmatrix} \begin{Bmatrix} u \\ N \end{Bmatrix}_A \quad (5)$$

and

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_C = \begin{bmatrix} \cos \theta & \frac{\ell}{EA} \frac{\sin \theta}{\theta} \\ -\mu \ell \omega^2 \frac{\sin \theta}{\theta} & \cos \theta \end{bmatrix} \begin{Bmatrix} u \\ N \end{Bmatrix}_B \quad (6)$$

Substituting eqn. (5) into eqn. (6) gives

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_C = \begin{bmatrix} \cos \theta & \frac{\ell}{EA} \frac{\sin \theta}{\theta} \\ -\mu \ell \omega^2 \frac{\sin \theta}{\theta} & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \frac{\ell}{EA} \frac{\sin \theta}{\theta} \\ -\mu \ell \omega^2 \frac{\sin \theta}{\theta} & \cos \theta \end{bmatrix} \begin{Bmatrix} u \\ N \end{Bmatrix}_A \quad (7)$$

Since A and C are the left and right end points of a rod of length  $2\ell$ , eqn. (7) states that the transfer matrix for longitudinal vibration of a rod of length  $2\ell$  is equal to the product of two transfer matrices for a rod of length  $\ell$ . To prove the validity of such a statement, first calculate the product of the two transfer matrices. Some mathematical manipulations give

$$\begin{bmatrix} \cos \theta & \frac{\ell}{EA} \frac{\sin \theta}{\theta} \\ -\mu \ell \omega^2 \frac{\sin \theta}{\theta} & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \frac{\ell}{EA} \frac{\sin \theta}{\theta} \\ -\mu \ell \omega^2 \frac{\sin \theta}{\theta} & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & 2 \frac{\ell}{EA} \frac{\sin 2\theta}{2\theta} \\ -2\mu \ell \omega^2 \frac{\sin 2\theta}{2\theta} & \cos 2\theta \end{bmatrix} \quad (8)$$

Since  $\theta = \ell \omega \sqrt{\frac{\rho}{E}}$  (refer to Appendix B),

$2\theta = 2\ell \omega \sqrt{\frac{\rho}{E}}$ , eqn. (7) can be obtained by substituting  $2\ell$  for  $\ell$  in the transfer matrix in eqn.

(3). Thus, the transfer matrix for longitudinal vibration of a rod of length  $2\ell$  is indeed equal to the product of two transfer matrices for a rod of length  $\ell$ .

In fact, the technique of transfer matrix multiplication can be applied to more complicated structures since the intermediate state vectors can be substituted successively to obtain the transfer matrix for the entire structure. It is this characteristic which makes the transfer matrix method a favorable approach in analyzing periodic structures. Thus, transfer matrices can be combined in such a way that intermediate state vectors can be eliminated.

For a specific problem, the excitation and the boundary conditions must be specified. When an excitation is applied to a structure, say at point (or station)  $p$ , the state vector becomes discontinuous at  $p$ . The problem is solved using the boundary conditions. This is illustrated in the next paragraph. For example, consider the rod shown in Fig. 4. The rod consists of four identical rod segments of length  $\ell$  and is loaded by a sinusoidal axial force of magnitude  $N_0$  at point  $C$ .

Now assume that the rod in Fig. 4 is broken at  $C$ . Fig. 5 shows the forces at the left end and the right end of the rod at  $C$ . Since displacements are continuous in crossing point  $C$  (that is,  $u_L$  at  $C = u_R$  at  $C$ ),

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_{C_R} = \begin{Bmatrix} u \\ N \end{Bmatrix}_{C_L} + \begin{Bmatrix} 0 \\ N \end{Bmatrix}_C \quad (9)$$

where  $C_R$  and  $C_L$  denote points just to the right and just to the left of  $C$ , respectively.

Let  $T(\ell)$  be the transfer matrix represented by the  $2 \times 2$  matrices in eqn. (3) when  $\ell$  in parentheses signifies that  $T$  is for the transfer matrix of a rod of length  $\ell$ . At  $C_L$ ,

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_{C_L} = T(\ell)T(\ell) \begin{Bmatrix} u \\ N \end{Bmatrix}_A \quad (10)$$

Since the rod is assumed to be a continuous member with no impedance mismatch, multiplying

$T(\ell)$  n times is equivalent to replacing  $\ell$  by  $n\ell$  in  $T$ . Thus, eqn. (10) can be rewritten as

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_{c_L} = T(2\ell) \begin{Bmatrix} u \\ N \end{Bmatrix}_A \quad (11)$$

Using eqn. (11), eqn. (9) becomes

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_{c_R} = T(2\ell) \begin{Bmatrix} u \\ N \end{Bmatrix}_A + \begin{Bmatrix} 0 \\ N_o \end{Bmatrix}_C \quad (12)$$

However, for point  $E$ ,

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_E = T(2\ell) \begin{Bmatrix} u \\ N \end{Bmatrix}_{c_R} \quad (13)$$

Combining eqn. (12) and eqn. (13) gives

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_E = T(2\ell) \left[ T(2\ell) \begin{Bmatrix} u \\ N \end{Bmatrix}_A + \begin{Bmatrix} 0 \\ N \end{Bmatrix}_C \right]$$

or

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_E = T(4\ell) \begin{Bmatrix} u \\ N \end{Bmatrix}_A + T(2\ell) \begin{Bmatrix} 0 \\ N_o \end{Bmatrix}_C \quad (14)$$

at the boundaries, since the the displacements are specified (that is,  $u = 0$ ), eqn. (14) can be written as

$$\begin{Bmatrix} 0 \\ N \end{Bmatrix}_E = T(4\ell) \begin{Bmatrix} 0 \\ N \end{Bmatrix}_A + T(2\ell) \begin{Bmatrix} 0 \\ N_o \end{Bmatrix}_C \quad (15)$$

Eqn. (15) can be solved to obtain the internal forces at  $A$  and  $E$ . In other words, the state vectors at  $A$  and  $E$  can be obtained by imposing the boundary conditions.

Since this is steady state vibration, the state vector at intermediate points in the rod can be obtained using either one of the boundary state vectors [8] (that is, the state vectors at  $A$  or  $E$ ). For example, for point  $B$ ,

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_B = T(\ell) \begin{Bmatrix} u \\ N \end{Bmatrix}_A \quad (16)$$

or

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_B = T^{-1}(3\ell) \begin{Bmatrix} u \\ N \end{Bmatrix}_E - T^{-1}(\ell) \begin{Bmatrix} 0 \\ N_o \end{Bmatrix}_C \quad (17)$$

and for point  $D$ ,

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_D = T(3\ell) \begin{Bmatrix} u \\ N \end{Bmatrix}_A + T(\ell) \begin{Bmatrix} 0 \\ N_o \end{Bmatrix}_C \quad (18)$$

or

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_D = T^{-1}(\ell) \begin{Bmatrix} u \\ N \end{Bmatrix}_E \quad (19)$$

where

$$T^{-1}(\ell) = \begin{bmatrix} \cos \theta & -\frac{\ell}{EA} \frac{\sin \theta}{\theta} \\ \mu \ell \omega^2 \frac{\sin \theta}{\theta} & \cos \theta \end{bmatrix}$$

is the transfer matrix in eqn. (3).

In addition, frequency response functions for other specific locations in a structure can be obtained through the use of transfer matrices [8]. With the frequency response functions known, random vibration can be considered. Moreover, impulse response functions can be generated from

the frequency response functions, whereby time histories of waves in a structure can be studied. [10]

The relationships between transfer matrices and propagation constants in periodic structures has been investigated in [9]. It has been found that at any particular frequency, the propagation constants corresponding to the waves in a structure are equal to the negative natural logarithms of the eigenvalues of the transfer matrix relating the state vectors at the two ends of the basic element constituting the periodic structure. Since propagation constants give information on attenuations, wave numbers and phase changes for wave propagation in structures (refer to Appendix A), the wave propagation characteristics can be readily obtained, via the propagation constant technique once the transfer matrix for a periodic unit is derived.

## APPLICATION OF THE TRANSFER MATRIX METHOD TO AN ELASTIC ROD WITH DISTRIBUTED MASS

Figure 6 shows the rod to be analyzed in this section. The rod has modulus of elasticity  $E$ , mass density  $\rho$  and cross-sectional area  $A$ . In addition, the rod is assumed to be made up of six identical rod segments, each of length  $\ell$ ; thus there are no impedance mismatches throughout the length of the rod.

### State Vectors

For longitudinal vibration in a rod, the state vector consists of a longitudinal displacement component  $u$  and an axial force component  $N$ . Figure 7 shows the sign convention to be used in this analysis. Thus,

$$z = \begin{Bmatrix} u \\ N \end{Bmatrix} \quad (20)$$

### Transfer Matrices

The transfer matrices (from left to right and from right to left) for longitudinal vibration in an elastic rod with distributed mass are derived in Appendix B. From eqn. (B11), the transfer matrix  $T$  which relates the state vector at the right end to the state vector at the left end of an elastic rod of length  $\ell$  is given as

$$T = \begin{bmatrix} \cos \theta & \frac{\ell}{EA} \frac{\sin \theta}{\theta} \\ -\mu \ell \omega^2 \frac{\sin \theta}{\theta} & \cos \theta \end{bmatrix} \quad (21)$$

and from eqn. (B16), the transfer matrix  $T^{-1}$  which relates the state vector at the left end to the state vector at the right end of an elastic rod is given as

$$T^{-1} = \begin{bmatrix} \cos \theta & -\frac{\ell}{EA} \frac{\sin \theta}{\theta} \\ \mu \ell \omega^2 \frac{\sin \theta}{\theta} & \cos \theta \end{bmatrix} \quad (22)$$

where the variables in eqn. (21) and (22) are defined in Appendix C.

Since the rod to be analyzed in this section has a length of  $6\ell$ , and since the rod is assumed to have constant material and geometric properties throughout its length with no impedance mismatches, it is convenient to divide the rod into six periodic units, with each periodic unit represented by a rod of length  $\ell$ . Thus eqns. (21) and (22) become the transfer matrices (from left to right and from right to left) of one periodic unit of the system.

#### Uses of the Transfer Matrix

With the transfer matrices for one periodic unit of the rod defined, the state vector for specific locations of the rod can be obtained. For example, referring to Fig. 6, the state vector at C going from left to right along the rod is given by

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_C = TT \begin{Bmatrix} u \\ N \end{Bmatrix}_A \quad (23)$$

or, which is the same, going from right to left along the rod,

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_C = T^{-1}T^{-1}T^{-1}T^{-1} \begin{Bmatrix} u \\ N \end{Bmatrix}_G \quad (24)$$

Since the rod has constant material and geometric properties throughout, eqns. (23) and (24) can be simplified to

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_C = T(2\ell) \begin{Bmatrix} u \\ N \end{Bmatrix}_A \quad (25)$$

and

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_C = T^{-1}(4\ell) \begin{Bmatrix} u \\ N \end{Bmatrix}_G \quad (26)$$

where the values in parentheses indicate the appropriate arguments in terms of  $\ell$  in the matrix  $T$ . Notice that if there are impedance mismatches in the rod, no such simplification can be made. For example, if section  $AB$  of the rod is made up of a material with modulus of elasticity  $2E$  instead of  $E$  as for the rest of the rod, eqn. (23) becomes

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_C = T \bar{T} \begin{Bmatrix} u \\ N \end{Bmatrix}_A \quad (27)$$

where  $\bar{T}$  represents the transfer matrix for section  $AB$  and is obtained by substituting  $2E$  for  $E$  in matrix  $T$ . Notice that eqn. (24) is not affected since section  $AB$  is not included.

Damping can also be included in the transfer matrix method. The transfer matrix from left to right for longitudinal vibration in an elastic rod with distributed mass and damping is derived in Appendix C. Assume that the rod in Figure 6 now has material damping which can be characterized by a viscous damping constant  $c$  and has no impedance mismatches. The state vector at  $C$  can still be obtained using eqn. (23) but with the transfer matrix  $T$  defined by eqn. (C 11) of Appendix C (instead of eqn. (22)).

The frequency response functions for specific locations in the rod are derived in Appendix F. Appendix F also contains the general approach whereby the problem of forced vibration in a rod can be treated. The rod is assumed to be excited by a sinusoidal axial force of magnitude  $N_o$  at  $E$  as shown in Figure 8. From eqn. (F 16),

$${}_N H_A^E(\omega) = - \frac{\cos 2\theta}{\cos 6\theta} \quad (28)$$

where the subscript  $A$  denotes the response location and the superscript  $E$  denotes the excitation location and the subscript  $N$  denotes an axial force response. From eqn. (F22),



$$H_B^E(\omega) = -\frac{\cos 2\theta}{\cos 6\theta} \begin{bmatrix} \frac{\ell}{EA} \frac{\sin \theta}{\theta} \\ \cos \theta \end{bmatrix} \quad (29)$$

From eqn. (F 23),

$$H_C^E(\omega) = -\frac{\cos 2\theta}{\cos 6\theta} \begin{bmatrix} \frac{2\ell}{EA} \frac{\sin 2\theta}{2\theta} \\ \cos 2\theta \end{bmatrix} \quad (30)$$

From eqn. (F 24),

$$H_D^E(\omega) = -\frac{\cos 2\theta}{\cos 6\theta} \begin{bmatrix} \frac{3\ell}{EA} \frac{\sin 3\theta}{3\theta} \\ \cos 3\theta \end{bmatrix} \quad (31)$$

From eqn. (F 25),

$$H_{E_L}^E(\omega) = -\frac{\cos 2\theta}{\cos 6\theta} \begin{bmatrix} \frac{4\ell}{EA} \frac{\sin 4\theta}{4\theta} \\ \cos 4\theta \end{bmatrix} \quad (32)$$

where the subscript  $E_L$  stands for the point just to the left of  $E$ . Also, from eqns. (F 31), (F 35) and (F 36),

$$H_G^E(\omega) = \frac{\frac{4\ell}{EA} \frac{\sin 4\theta}{4\theta}}{\cos 6\theta} \quad (33)$$

where the subscript  $u$  denotes the displacement response,

$$H_F^E(\omega) = \frac{\frac{4\ell}{EA} \frac{\sin 4\theta}{4\theta}}{\cos 6\theta} \begin{bmatrix} \cos \theta \\ \mu \ell \omega^2 \frac{\sin \theta}{\theta} \end{bmatrix} \quad (34)$$

and

$$H_{E_R}^E(\omega) = \frac{\frac{4\ell}{EA} \frac{\sin 4\theta}{4\theta}}{\cos 6\theta} \left[ \frac{\cos 2\theta}{2\mu\ell\omega^2} \frac{\sin 2\theta}{2\theta} \right] \quad (35)$$

where the subscript  $E_R$  stands for the point just to the right of  $E$ .

The impulse response function at  $B$  is also generated in Appendix F. Due to the simple nature of  $H_B^E(\omega)$ , the impulse response function is obtained in closed form. From eqn. (F 44), the impulse response for the force at  $B$  is given as

$$\begin{aligned} h_B^E(t) = \frac{1}{2} \sum_{k=-\infty}^{\infty} \{ & -\delta[t - (3+24k)\ell\sqrt{\frac{\rho}{E}}] - \delta[t - (5+24k)\ell\sqrt{\frac{\rho}{E}}] \\ & -\delta[t - (7+24k)\ell\sqrt{\frac{\rho}{E}}] - \delta[t - (9+24k)\ell\sqrt{\frac{\rho}{E}}] \\ & +\delta[t - (15+24k)\ell\sqrt{\frac{\rho}{E}}] + \delta[t - (17+24k)\ell\sqrt{\frac{\rho}{E}}] \\ & +\delta[t - (19+24k)\ell\sqrt{\frac{\rho}{E}}] + \delta[t - (21+24k)\ell\sqrt{\frac{\rho}{E}}] \} \end{aligned} \quad (36)$$

For an impulse excitation at  $E$ , eqn. (36) gives the time-history of the response to be expected at  $B$ . Two points are of particular interest here. First, the fraction  $\frac{1}{2}$  in front of the summation sign signifies that the amplitudes of the responses at  $B$  with respect to time is always one half of a delta function. This is expected because, due to symmetry in the rod, the axial force excitation is split up into two equal and opposite going waves traveling along the rod. Second, notice that the term  $\ell\sqrt{\frac{\rho}{E}}$  has the unit of time. In fact, it represents the time required for the axial wave to travel a distance  $\ell$  along the rod.

In addition, at any particular frequency, the propagation constant can be obtained as the negative logarithms of the eigen values of transfer matrix  $T$ . Since  $T$  is a  $2 \times 2$  matrix, there is only one pair of equal and opposite propagation constants, corresponding to two opposite and identical waves as noted earlier [9].

#### APPLICATION OF THE TRANSFER MATRIX METHOD TO A 3-BAY PLANAR LATTICE STRUCTURE

Figure 9 shows the 3-bay planar lattice structure to be analyzed in this section. For simplicity, assume that the cross-sectional dimensions in the bars are small compared with the lengths, and that the structure is made up of identical horizontal and vertical bar elements throughout. Furthermore, the Timoshenko beam model is used for the bars such that the effects of shear deflection and rotary inertia are included in the analysis.

##### State Vector

Recall that the vibration and wave propagation in a structure are characterized by a state vector  $z$ . In the case of the planar structure shown in Figure 9, for each bar, the state vector  $z$  consists of three displacement component and three internal force components. The three displacement components are  $u$ ,  $w$  and  $\psi$ , where  $u$  is the longitudinal displacement,  $w$  is the transverse displacement and  $\psi$  is the rotation of the cross-section. The three force components are  $M$ ,  $V$  and  $N$ , where  $M$  is the moment,  $V$  is the shear force and  $N$  is the axial force. Figure 10 shows the sign convention for the forces and the displacements. Thus

$$z = \begin{Bmatrix} d \\ p \end{Bmatrix} \quad (37)$$

where  $d$  is the displacement vector and  $p$  is the force vector such that

$$d = \begin{Bmatrix} u \\ -w \\ \psi \end{Bmatrix}$$

and

$$p = \begin{Bmatrix} M \\ V \\ N \end{Bmatrix}$$

In analyzing the structure shown in Figure 9, state vectors corresponding to both main members I and II are needed to describe the vibration characteristics of the structure. The reason for this is due to the choice of the transfer matrices and will become apparent in the next section. Thus, for a particular section in the structure, the state vector  $Z$  of interest is

$$Z = \begin{Bmatrix} d_I \\ d_{II} \\ p_{II} \\ p_I \end{Bmatrix} \quad (38)$$

where the subscripts denote member numbers. Notice that  $Z$  is used to differentiate the state vector at a section in a structure from that of  $z$ , which represents the state vector at a point in the structure.

#### Transfer Matrices

Two transfer matrices are involved here. The first transfer matrix  $X_1$ , involves the transfer of state vectors in two bars, each of length  $\ell$  in main members I and II. For example, members 12 and 12' together are represented by such a matrix. The second transfer matrix  $X_2$  involves the transfer of state vectors across the junctions. The members which join main members I and II constitute such a matrix. For example, member 11 constitutes such a matrix. Figure 11 shows the 3-bay planar structure which has been sectioned into its constituent parts responsible for transfer matrices  $X_1$  and  $X_2$ . The subscripts  $R$  and  $L$  are used to denote points which are just to the right and points which are just to the left, respectively, of junctions which join main members I and II.

Transfer matrices  $X_1$  and  $X_2$  are derived in Appendix G. From eqn. (G4),

$$X_1 = \begin{bmatrix} C_1 & 0 & 0 & C_2 \\ 0 & C_1 & C_2 & 0 \\ 0 & C_3 & C_4 & 0 \\ C_3 & 0 & 0 & C_4 \end{bmatrix} \quad (39)$$

and from eqn (G 28),

$$X_2 = \begin{bmatrix} I & & 0 & 0 \\ 0 & & 0 & 0 \\ G_4 C_2^{-1} G_1 & -G_4 C_2^{-1} C_1 G_3 & I & 0 \\ G_2 C_4 C_2^{-1} G_1 & G_2 (C_3 - C_4 C_2^{-1} C_1) G_3 & 0 & I \end{bmatrix} \quad (40)$$

The variables used in eqns. (39) and (40), namely  $C_1, C_2, C_3, C_4, G_1, G_2, G_3$  and  $G_4$ , are defined in Appendix G.

### Uses of the Transfer Matrices

With transfer matrices  $X_1$  and  $X_2$  defined, the transfer matrix  $T$  for one periodic unit of the planar structure can be obtained. Figure 12 shows the planar structure which has been sectioned into four basic segments, namely,  $A_L O O' A_L' A_L$ ,  $A_R A_R' 1' B_L' B_L 1 A_R$ ,  $B_R B_R' 2' C_L' C_L 2 B_R$  and  $C_R C_R' 3' 3 C_R$ .  $A_R A_R' 1' B_L' B_L 1 A_R$  and  $B_R B_R' 2' C_L' C_L 2 B_R$  each define a basic periodic unit while  $A_L O O' A_L' A_L$  defines a left half-unit and  $C_R C_R' 3' 3 C_R$  defines a right half-unit.

Consider first section  $A_R A_R' 1' B_L' B_L 1 A_R$  in Figure 12. For section  $1_L - 1_L'$ ,

$$\begin{pmatrix} d_I \\ d_{II} \\ p_{II} \\ p_I \end{pmatrix}_{1_L - 1_L'} = X_1^{1/2} \begin{pmatrix} d_I \\ d_{II} \\ p_{II} \\ p_I \end{pmatrix}_{A_R - A_R'} \quad (41)$$

where  $X_1^{1/2}$  is the transfer matrix which includes lengths of  $\ell/2$  in both main members I and II, and where  $1_L$  and  $1_L'$  are points just to the left of points 1 and  $1'$ , respectively. Similarly, for section  $1_R - 1_R'$ , where  $1_R$  and  $1_R'$  are points just to the right of points 1 and  $1'$ , respectively,

$$\begin{pmatrix} d_I \\ d_{II} \\ p_{II} \\ p_I \end{pmatrix}_{1_R - 1_R'} = X_2 \begin{pmatrix} d_I \\ d_{II} \\ p_{II} \\ p_I \end{pmatrix}_{1_L - 1_L'} \quad (42)$$

and for section  $3_L - 3_L'$ ,

$$\begin{pmatrix} d_I \\ d_{II} \\ p_{II} \\ p_I \end{pmatrix}_{B_L - B_L'} = X_1^{1/2} \begin{pmatrix} d_I \\ d_{II} \\ p_{II} \\ p_I \end{pmatrix}_{1_R - 1_R'} \quad (43)$$

Substituting eqns. (41) and (42) into eqn. (43) gives

$$\begin{pmatrix} d_I \\ d_{II} \\ p_{II} \\ p_I \end{pmatrix}_{B_L - B_L'} = X_1^{1/2} X_2 X_1^{1/2} \begin{pmatrix} d_I \\ d_{II} \\ p_{II} \\ p_I \end{pmatrix}_{A_R - A_R'} \quad (44)$$

Since section  $A_R A_R' 1' B_L' B_L 1 A_R$  is representative of a periodic unit, the transfer matrix  $T$  for one periodic unit is obtained from eqn. (44) as

$$T = X_1^{1/2} X_2 X_1^{1/2} \quad (45)$$

Similarly, the transfer matrix for  $A_L 00' A_L' A_L$ , denoted by  $T^{-1/2}$  is given by

$$T^{-1/2} = X_1^{1/2} X_2 \quad (46)$$

and the transfer matrix for  $C_R C_R' 3' 3 C_R$ , denoted by  $T^{1/2}$ , is given by

$$T^{1/2} = X_2 X_1^{1/2} \quad (47)$$

Notice that the transfer matrices  $T$ ,  $T^{-1/2}$  and  $T^{1/2}$  are all transfer matrices which relate state vectors from left to right. In addition, the order of the transfer matrices on the right hand sides of eqns. (45), (46) and (47) are important.

After the transfer matrices  $T$ ,  $T^{-1/2}$ ,  $T^{1/2}$  are defined, the state vectors at specific sections of the planar structure can be obtained. For example for section  $C-C'$ ,

$$\begin{Bmatrix} d_I \\ d_{II} \\ p_{II} \\ p_I \end{Bmatrix}_{C-C'} = T T T^{-1/2} \begin{Bmatrix} d_I \\ d_{II} \\ p_{II} \\ p_I \end{Bmatrix}_{O_L-O_L'} \quad (48)$$

As another example, for section  $3-3'$ ,

$$\begin{Bmatrix} d_I \\ d_{II} \\ p_{II} \\ p_I \end{Bmatrix}_{3-3'} = T^{-1/2} T T T^{-1/2} \begin{Bmatrix} d_I \\ d_{II} \\ p_{II} \\ p_I \end{Bmatrix}_{O_L-O_L'} \quad (49)$$

The frequency response functions for specific sections of the planar structure are derived in Appendix H. The planar structure is assumed to be excited by sinusoidal shear forces at  $C$  and  $C'$ . In addition to deriving the frequency response functions for the planar structure, Appendix H also gives the general approach whereby the problem of forced vibration in a planar structure can be treated. From eqn. (H9) and eqn. (H10),

$$H_A^{C-C'}(\omega) = - \begin{bmatrix} b_{11}' & b_{12}' \\ b_{41}' & b_{42}' \end{bmatrix} \begin{bmatrix} k_{31} & k_{32} \\ k_{41} & k_{42} \end{bmatrix}^{-1} \begin{bmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{bmatrix} \quad (50)$$

and

$$H_A^{c-c'}(\omega) = - \begin{bmatrix} b_{21}^c & b_{22}^c \\ b_{31}^c & b_{32}^c \end{bmatrix} \begin{bmatrix} k_{31} & k_{32} \\ k_{41} & k_{42} \end{bmatrix}^{-1} \begin{bmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{bmatrix} \quad (51)$$

where the superscripts denote the excitation location and the subscripts denote the response locations.

From eqn. (H 11) and eqn. (H 12),

$$H_B^{c-c'}(\omega) = - \begin{bmatrix} b_{11}^c & b_{12}^c \\ b_{41}^c & b_{22}^c \end{bmatrix} \begin{bmatrix} k_{31} & k_{32} \\ k_{41} & k_{42} \end{bmatrix}^{-1} \begin{bmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{bmatrix} \quad (52)$$

and

$$H_B^{1c-c'}(\omega) = - \begin{bmatrix} b_{21}^c & b_{22}^c \\ b_{31}^c & b_{32}^c \end{bmatrix} \begin{bmatrix} k_{31} & k_{32} \\ k_{41} & k_{42} \end{bmatrix}^{-1} \begin{bmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{bmatrix} \quad (53)$$

From eqns. (H 30), (H 32) and (H 31),

$$H_4^{c-c'}(\omega) = -T^{1/2} \begin{bmatrix} G_1' & 0 \\ 0 & G_2' \end{bmatrix} \begin{bmatrix} b_{21}^{IV} & b_{22}^{IV} \\ b_{31}^{IV} & b_{32}^{IV} \end{bmatrix} \begin{bmatrix} k_{31} & k_{32} \\ k_{41} & k_{42} \end{bmatrix}^{-1} \begin{bmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{bmatrix} \quad (54)$$

$$H_5^{c-c'}(\omega) = -T^{1/2} \begin{bmatrix} G_1' & 0 \\ 0 & G_2' \end{bmatrix} \begin{bmatrix} b_{21}^{VV} & b_{22}^{VV} \\ b_{31}^{VV} & b_{32}^{VV} \end{bmatrix} \begin{bmatrix} k_{31} & k_{32} \\ k_{41} & k_{42} \end{bmatrix}^{-1} \begin{bmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{bmatrix} \quad (55)$$

$$H_5^{c-c'} = -T^{1/2} \begin{bmatrix} G_1' & 0 \\ 0 & G_2' \end{bmatrix} \begin{bmatrix} b_{21}^{VV} & b_{22}^{VV} \\ b_{31}^{VV} & b_{32}^{VV} \end{bmatrix} \begin{bmatrix} k_{31} & k_{32} \\ k_{41} & k_{42} \end{bmatrix}^{-1} \begin{bmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{bmatrix} \quad (56)$$

The variables used in eqns (50) through (56) are defined in Appendix H.

The impulse response functions can be obtained by using the inverse Fourier Transform as [10]

$$h_i^a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_i^a(\omega) e^{i\omega t} d\omega \quad (57)$$

where the subscript  $q$  denotes the response location of interest and the superscript  $a$  denotes the excitation location.

In addition, at any particular frequency, the propagation constants can be found by obtaining the negative logarithms of the eigenvalues of the transfer matrix  $T$  for one periodic unit of the structure. Notice that since  $T$  is a  $12 \times 12$  matrix, there are six pairs of opposite identical propagation constants, as mentioned earlier.



# APPLICATION OF THE TRANSFER MATRIX METHOD TO A TETRAHEDRAL TRUSS

Figure 14 shows a tetrahedral truss which is used in this section. The tetrahedral truss is assumed to consist of identical elastic bars with distributed mass and circular cross sections, each of length  $\ell$ . Each connecting bar has modulus of elasticity  $E$ , mass density  $\rho$ , shear modulus  $G$ , cross-sectional area  $A$ , second moment of area inertia about the x or z axis  $J$ , second moment of area inertia about the y axis  $J_y$  and radius of gyration about the x or z axis  $i$ . Furthermore, the across-sectional dimensions of each bar are assumed to be small compared with its length.

## State Vectors

For each connecting bar in the tetrahedral truss shown in Figure 1, the state vector  $z$  consists of six force components and six displacement components. The six force components are  $M_y, M_z, T, V_y, V_z$ , and  $N$ , where  $M_y$  is the moment about the y-axis,  $M_z$  is the moment about the z-axis,  $T$  is the torque about the x-axis,  $V_y$  is the shear force along the z direction,  $V_z$  is the shear force along the y direction and  $N$  is the axial force. The six displacement components include  $u, v, w, \phi, \psi$  and  $\theta$ , where  $u$  is the longitudinal displacement,  $v$  is the lateral displacement in the y direction,  $w$  is the lateral displacement in the z direction and  $\phi, \psi$  and  $\theta$  are rotations of the cross section about the z, y and x axes, respectively. Thus, for each connecting bar,

$$z = \begin{Bmatrix} d \\ p \end{Bmatrix} \quad (53)$$

where  $d$  is the displacement vector such that

$$d = \begin{Bmatrix} u \\ v \\ w \\ \phi \\ \psi \\ \theta \end{Bmatrix}$$

and where  $p$  is the force vector such that

$$p = \begin{pmatrix} M_x \\ M_y \\ T \\ V_x \\ V_y \\ N \end{pmatrix}$$

Figure 15 shows the sign convention for the force and displacement variables.

In the analysis of the tetrahedral truss, for a particular station (or section), the state vectors in main members I through IV are needed. The reason for this is due to the choice of the transfer matrices and will become apparent in the next section. Thus, for each particular station,

$$Z = \begin{pmatrix} d_I \\ d_{II} \\ d_{III} \\ d_{IV} \\ p_{IV} \\ p_{III} \\ p_{II} \\ p_I \end{pmatrix} \quad (59)$$

where the subscripts I, II, III or IV denotes the member number for which d or p is defined (refer to Figure 14).

#### Transfer Matrices

In the analysis of the tetrahedral truss, three transfer matrices, namely,  $V_1, V_2$  and  $V_3$  are required. Figure 16 shows a segment of the tetrahedral truss and Figure 17 shows the segment in Figure 16 which is sectioned along planes parallel to the yz plane into the substructures responsible for transfer matrices  $V_1, V_2$  and  $V_3$ . Referring to Figure 17, the sections are made by cutting the periodic unit along planes  $GA_RID_R, E_RK_RF_RL_R$  (or  $E_LK_LF_LL_L$ ) and  $HB_LJC_L$ . The subscripts L and R used for points E, A, D, F, K, L, B and C denote points just left and right of these points, respectively. Thus, the first transfer matrix  $V_1$  represents the transfer of state vectors in members which join main members I and III. For example, members AD and BL each constitutes a transfer matrix  $V_1$ . The second transfer matrix  $V_2$  represents the transfer of state vectors in four bars, each of length  $\ell/2$  in main members I through IV together with members

which connect member I to member II, member III to member II, member I to member IV, member III to member IV. Referring to Figure 17, this includes members  $A_R K_L$ ,  $I F_L$ ,  $D_R L_L$ ,  $G E_L$ ,  $A_R f_L$ ,  $D_R F_L$ ,  $A_R E_L$  and  $D_R E_L$ . The third transfer matrix  $V_3$  is responsible for the transfer of state vectors in four bars, each of length  $\ell/2$  in members I through IV together with member which connect member II to member I, member II to member III, member IV to member I and Member IV to member III. Referring to Figure 17, this includes members  $K_R B_L$ ,  $F_R J$ ,  $L_R C_L$ ,  $E_R H$ ,  $F_R B_L$ ,  $F_R C_L$ ,  $E_R B_L$  and  $E_R C_L$ .

The transfer matrices  $V_1$ ,  $V_2$  and  $V_3$  are derived in Appendix I. From eqn. (I23), the transfer matrix  $V_1$  is given as

$$V_1 = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ G_2(C_3 - C_4 C_2^{-1} C_1) G_3 & 0 & G_2 C_4 C_2^{-1} G_1 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ -G_4 C_2^{-1} C_1 G_3 & 0 & G_4 G_2^{-1} G_1 & 0 & 0 & 0 & 0 & I \end{bmatrix} \quad (60)$$

From eqn (I 64), the transfer matrix  $V_2$  is given as

$$V_2 = \begin{bmatrix} C_1' & 0 & 0 & 0 & 0 & 0 & 0 & C_2' \\ 0 & C_1' & 0 & 0 & 0 & 0 & C_2' & 0 \\ 0 & 0 & C_3' & 0 & 0 & C_2' & 0 & 0 \\ 0 & 0 & 0 & C_3' & C_2' & 0 & 0 & 0 \\ D_{1,2} & 0 & D_{1,3} & D_{1,1} + D_{1,3} C_3' & C_4 & 0 & 0 & 0 \\ 0 & D_5 & D_5 + D_{1,4} - C_3' & D_{1,3} & 0 & C_4' & 0 & 0 \\ 4 & D_3 - D_2 - C_3' & D_3 & 0 & 0 & 0 & C_4' & 0 \\ D_2 + D_{1,2} - C_3' & D_1 & 0 & D_{1,3} & 0 & 0 & 0 & C_4' \end{bmatrix} \quad (61)$$

From eqn. (I 65), the transfer matrix  $V_3$  is given as

$$V_3 = \begin{pmatrix} C_1' & 0 & 0 & 0 & 0 & 0 & 0 & C_2' \\ 0 & C_1' & 0 & 0 & 0 & 0 & C_2' & 0 \\ 0 & 0 & C_1' & 0 & 0 & C_2' & 0 & 0 \\ 0 & 0 & 0 & C_1' & C_2' & 0 & 0 & 0 \\ E_9 & 0 & E_{13} & E_{10}+E_{14}+C_3' & C_4' & 0 & 0 & 0 \\ 0 & E_8 & E_7+E_{15}+C_3' & E_{10} & 0 & C_4' & 0 & 0 \\ E_1 & E_2+E_6+C_3' & E_5 & 0 & 0 & 0 & C_4' & 0 \\ E_3+E_{11}+C_3' & E_4 & 0 & E_{12} & 0 & 0 & 0 & C_4' \end{pmatrix} \quad (62)$$

The variable used in the transfer matrices in eqns. (60) through (62) are defined in Appendix I.

#### Uses of the Transfer Matrices

With transfer matrices  $V_1, V_2$  and  $V_3$  defined, the transfer matrix  $T$  for one periodic unit of the tetrahedral truss can be obtained. Figure 18 shows the tetrahedral truss sectioned into four basic segments. Referring to Figure 18, segment 1 defines a left half periodic unit, segment 2 and 3 each defines a periodic unit and segment 4 defines a right half periodic unit. Following the same procedure used for the 3-bay planar structure in the last chapter, the transfer matrix for the left half periodic unit, denoted by  $T^{-1/2}$ , is given as

$$T^{-1/2} = V_2 V_1 \quad (63)$$

Similarly, the transfer matrix for a periodic unit, denoted by  $T$ , is given by

$$T = V_2 V_1 V_3 \quad (64)$$

and the transfer matrix for the right half periodic unit, denoted by  $T^{1/2}$ , is given by

$$T^{1/2} = V_1 V_3 \quad (65)$$

### NUMERICAL EXAMPLE

To illustrate the application of the transfer matrix method in the analysis of wave propagation in periodic structures the case of a one-dimensional elastic rod is investigated. The frequency response functions for specific locations for the longitudinal vibration in an elastic rod are obtained. Based on these results, the impulse response function for a location in the rod is generated, whereby the wave propagation characteristics for both square pulse excitation and a triangular pulse excitation in the rod are studied.

Figure 19 shows the rod to be investigated in this example. The rod is assumed to consist of six identical rod elements, each of length  $\ell$ , with no impedance mismatches and no material damping throughout. The material and geometric properties in the rod are given as

$$E = 7.46 \times 10^{10} \text{ Pa} \quad (10.8 \times 10^6 \text{ psi})$$

$$A = 6.29 \times 10^{-5} \text{ m}^2 \quad (9.75 \times 10^{-2} \text{ in}^2)$$

$$\rho = 2767 \text{ kg/m}^3 \quad (0.1 \text{ lbf/in}^3)$$

$$\ell = 0.25 \text{ m} \quad (10 \text{ in})$$

where  $E$  is the elastic modulus,  $A$  is the cross-sectional area and  $\rho$  is the mass density.

The frequency response functions for specific locations in the rod due to a sinusoidal axial force excitation at point E have been generated in eqns. (28) through (35). Based on these results, a basic computer program, named PROG1.BAS (refer to Appendix J), is written to obtain the frequency response functions numerically. For demonstration purposes, a plot of the frequency response of the force at B due to excitation at E versus frequency is shown in Figure 20. Jumps in the values of the frequency response function in Figure 20 signify resonance conditions. For comparison, the first nine natural frequencies for the rod are tabulated in Table 1 [14]. Modes 2, 5 and 8 in Table 1 are not shown in Fig. 20 because the excitation point becomes a nodal point at

such frequencies. However, since PROG1.BAS calculates the frequency response function at  $B$  at discrete frequency intervals, the values for the response at resonances are truncated by the program and do not reach infinity. The impulse force response function at  $B$  due to a unit force excitation at  $E$  is generated in eqn. (F44) and is given by

$$\begin{aligned} n h_B^E(t) = \frac{1}{2} \sum_{k=0}^{\infty} \{ & -\delta[t - (3+24k)\ell\sqrt{\frac{\rho}{E}}] - \delta[t - (5+24k)\ell\sqrt{\frac{\rho}{E}}] \\ & - \delta[t - (7+24k)\ell\sqrt{\frac{\rho}{E}}] - \delta[t - (9+24k)\ell\sqrt{\frac{\rho}{E}}] \\ & + \delta[t - (15+24k)\ell\sqrt{\frac{\rho}{E}}] + \delta[t - (17+24k)\ell\sqrt{\frac{\rho}{E}}] \\ & + \delta[t - (19+24k)\ell\sqrt{\frac{\rho}{E}}] + \delta[t - (21+24k)\ell\sqrt{\frac{\rho}{E}}] \} \end{aligned} \quad (66)$$

Based on eqn. (66), a plot of the impulse force response at  $B$  versus elapsed time intervals is shown in Fig. 21. A time interval of  $\ell\sqrt{\frac{\rho}{E}}$ , which is numerically equal to  $4.95 \times 10^{-5}$  sec is used. Note that  $\ell\sqrt{\frac{\rho}{E}}$  is the time required for the impulse excitation to travel a distance  $\ell$  along the rod. Due to symmetry conditions, the impulse excitation is divided into equal and opposite going waves, each having magnitude equal to half a Dirac delta function, which explains the amplitude of the response.

Physically, eqn. (66) states that if a unit impulse force is applied at  $E$  at time zero, a force response of amplitude equal to half a Dirac delta function will be observed after  $3\ell\sqrt{\frac{\rho}{E}}$ ,  $5\ell\sqrt{\frac{\rho}{E}}$ , and so on. To understand this further, refer to Fig. 19. At time zero, a unit impulse force of positive magnitude is applied at  $E$ . Due to symmetry conditions, this force is divided into two equal and opposite going waves, each having a magnitude equal to half of the applied amplitude. After three time steps of  $\ell\sqrt{\frac{\rho}{E}}$ , the left-going wave reaches  $B$ . Since  $B$  is now under compression, it experiences a force of negative magnitude. This explains the first response shown in

Fig. 21. Now, upon reaching  $B$ , the left going-wave travels further along the rod until it reaches the boundary at  $A$  after an additional  $\ell\sqrt{\frac{\rho}{E}}$ . The wave is then reflected at boundary  $A$ . However, since boundary  $A$  is fixed, there is no sign change in the wave due to reflection and after another time step of  $\ell\sqrt{\frac{\rho}{E}}$ , it reaches  $B$  again, and now  $B$  experiences another force of negative magnitude. This explains the response at  $5\ell\sqrt{\frac{\rho}{E}}$  in Fig. 21. Now consider the right-going wave at  $E$ . After two time steps of  $\ell\sqrt{\frac{\rho}{E}}$  from time zero, the right-going wave reaches boundary  $G$  and is reflected. However, since boundary  $G$  is a free boundary, the magnitude of the wave changes from positive to negative. After another five time steps of  $\ell\sqrt{\frac{\rho}{E}}$ , this negative wave reaches  $B$ . Now  $B$  experiences a force of negative magnitude. This explains the response at  $7\ell\sqrt{\frac{\rho}{E}}$  in Fig. 21. The negative wave then travels further along the rod and after  $\ell\sqrt{\frac{\rho}{E}}$ , it reaches boundary  $A$  and is reflected. Since boundary  $A$  is fixed, there is no change in the sign of the wave due to reflection and after another  $\ell\sqrt{\frac{\rho}{E}}$ , it reaches  $B$  and  $B$  again experiences a negative force. This explains the response at  $9\ell\sqrt{\frac{\rho}{E}}$  in Fig. 21. Following the same procedure, each individual response at  $B$  as given by eqn. (66) can be explained.

Once the impulse response function is obtained for a particular location, the response due to other forms of excitation can be obtained using the relation [10]

$${}_a y_p^q(t) = \int_{-\infty}^{\infty} {}_a h_p^q(t-\tau) x(\tau) d(\tau) \quad (67)$$

where  ${}_a y_p^q(t)$  is the response function and  $x(\tau)$  is the excitation function. In addition, for the functions  ${}_a y_p^q(t)$  and  ${}_a h_p^q(t-\tau)$ , the subscript  $a$  denotes the type of response (for example, axial force), the subscript  $p$  denotes the response location and the subscript  $q$  denotes the excitation

location.

For the present case, if the rod is excited by an excitation function  $x(\tau)$ , the response  $y_B^E(\tau)$ , can be obtained by substituting  $h_B^E(\tau)$  from eqn. (66) into eqn. (67). Because  $h_B^E(\tau)$  in eqn. (66) consists of eight Dirac delta functions, each Dirac delta function has to be integrated separately. For example, consider the term  $\delta[t - (3+24k)\ell\sqrt{\frac{\rho}{E}}]$  in eqn. (66). The response  $y_B^E(t)$  due to this function is given by

$$y_B^E(t) = \int_{-\infty}^{\infty} \delta[t - (3+24k)\ell\sqrt{\frac{\rho}{E}} - \tau] x(\tau) d\tau \quad (68)$$

where the superscript 1 denotes the response is due to the first Dirac delta function in eqn. (66).

Using the relation [10]

$$\int_{-\infty}^{\infty} \delta(T - \tau) f(\tau) d\tau = f(T) \quad (69)$$

where  $f(\tau)$  is any function, eqn. (68) becomes

$$y_B^E(t) = x[t - (3+24k)\ell\sqrt{\frac{\rho}{E}}] \quad (70)$$

The same procedure can be applied to the other Dirac delta functions in eqn. (66) to obtain the total response  $y_B^E(t)$  from the integral in eqn. (67) as

$$\begin{aligned} y_B^E(t) = \frac{1}{2} \sum_{k=-\infty}^{\infty} \{ & -x[t - (3+24k)\ell\sqrt{\frac{\rho}{E}}] - x[t - (5+24k)\ell\sqrt{\frac{\rho}{E}}] \\ & -x[t - (7+24k)\ell\sqrt{\frac{\rho}{E}}] - x[t - (9+24k)\ell\sqrt{\frac{\rho}{E}}] \\ & -x[t - (15+24k)\ell\sqrt{\frac{\rho}{E}}] + x[t - (17+24k)\ell\sqrt{\frac{\rho}{E}}] \\ & +x[t - (19+24k)\ell\sqrt{\frac{\rho}{E}}] + x[t - (21+24k)\ell\sqrt{\frac{\rho}{E}}] \} \end{aligned} \quad (71)$$

By using eqn. (71), the response at  $E$  can be obtained. two types of excitations are considered here to illustrate the approach.

First consider the rod to be excited at  $E$  by a square pulse of unit amplitude and of



duration  $\ell \sqrt{\frac{\rho}{E}}$ . Thus, the excitation  $X_1(\tau)$  is given as

$$X_1(\tau) = \begin{cases} 1 & 0 \leq \tau \leq \ell \sqrt{\frac{\rho}{E}} \\ 0 & \text{otherwise} \end{cases} \quad (72)$$

If eqn. (72) is substituted into eqn. (71), the response  $y_B^E(t)$  is obtained as

$$\begin{aligned} y_B^E(t) = \frac{1}{2} \sum_{k=0}^{\infty} \{ & -X_1[t - (3+24k)\ell \sqrt{\frac{\rho}{E}}] - X_1[t - (5+24k)\ell \sqrt{\frac{\rho}{E}}] \\ & - X_1[t - (7+24k)\ell \sqrt{\frac{\rho}{E}}] - X_1[t - (9+24k)\ell \sqrt{\frac{\rho}{E}}] \\ & + X_1[t - (15+24k)\ell \sqrt{\frac{\rho}{E}}] + X_1[t - (17+24k)\ell \sqrt{\frac{\rho}{E}}] \\ & + X_1[t - (19+24k)\ell \sqrt{\frac{\rho}{E}}] + X_1[t - (21+24k)\ell \sqrt{\frac{\rho}{E}}] \} \end{aligned} \quad (73)$$

Figure 22 shows a plot of the first few responses. With respect to time intervals of  $\ell \sqrt{\frac{\rho}{E}}$  each.

Similarly, if the rod is now excited at E by a triangular pulse of duration  $\ell \sqrt{\frac{\rho}{E}}$  and a peak amplitude of unity such that  $X_2(\tau)$  is given as

$$X_2(\tau) = \begin{cases} \frac{\tau}{\ell} \sqrt{\frac{E}{\rho}} & 0 \leq \tau < \frac{\ell}{2} \sqrt{\frac{\rho}{E}} \\ (\ell \sqrt{\frac{\rho}{E}} - \tau) \frac{1}{\ell} \sqrt{\frac{E}{\rho}} & \frac{\ell}{2} \sqrt{\frac{\rho}{E}} \leq \tau \leq \ell \sqrt{\frac{\rho}{E}} \\ 0 & \text{otherwise} \end{cases} \quad (74)$$

If eqn. (74) is substituted into eqn. (71) the response  $y_B^E(t)$  is given as

$$\begin{aligned} y_B^E(t) = \frac{1}{4} \sum_{k=0}^{\infty} \{ & -X_2[t - (3+24k)\ell \sqrt{\frac{\rho}{E}}] - X_2[t - (5+24k)\ell \sqrt{\frac{\rho}{E}}] \\ & - X_2[t - (7+24k)\ell \sqrt{\frac{\rho}{E}}] - X_2[t - (9+24k)\ell \sqrt{\frac{\rho}{E}}] \\ & + X_2[t - (15+24k)\ell \sqrt{\frac{\rho}{E}}] + X_2[t - (17+24k)\ell \sqrt{\frac{\rho}{E}}] \\ & + X_2[t - (19+24k)\ell \sqrt{\frac{\rho}{E}}] + X_2[t - (21+24k)\ell \sqrt{\frac{\rho}{E}}] \} \end{aligned} \quad (75)$$

Figure 23 shows a plot of the first few responses at B, with respect to time intervals of  $\ell \sqrt{\frac{\rho}{E}}$

each.

The explanation of the physical implications of both eqns. (73) and (75) are very similar to that of eqn. (66) and thus are not explained further.

## CONCLUSIONS AND RECOMMENDATIONS

The transfer matrix method is observed to be a simple and straightforward approach in analyzing wave propagation and vibration in periodic structures. With the recent advances in the field of transfer matrices, the transfer matrix method becomes even more favorable. The transfer matrix method permits a simple treatment of periodic units with complicated configurations. Moreover, a matrix formation is most suitable for periodic structures of finite total length since the imposition of the boundary conditions is straightforward as is observed in the analyses given for the one-dimensional elastic rod and the 3-bay planar lattice structure. Some common properties of transfer matrices are outlined in Appendix K, which may serve to simplify calculations in applying the transfer matrix method. The transfer matrices for longitudinal vibration in an elastic rod, as given in eqns. (B 11) and (B 16) in Appendix B and the transfer matrix for flexural vibration in a Timoshenko Beam as given in eqn. (D 16) of Appendix D are used in Appendix K to demonstrate the properties outlined.

In addition, with the aid of computers, the transfer matrix method can be applied with little difficulty once the transfer matrix for a periodic unit is formulated, as is demonstrated in the numerical example.

However, there is one setback in the transfer matrix method. If a periodic structure to be analyzed consists of a large number of repeating periodic units, the numerical application of the transfer matrix method requires multiplication of a large number of transfer matrices together, the product of which may become enormously large and difficult to handle, even with the aid of a computer. In addition, if the elements in a transfer matrix are frequency dependent, they become larger and larger with increasing frequencies. The accuracy of the transfer matrix method may be significantly reduced due to the operations of large numbers. This is demonstrated by the inaccuracies in obtaining the determinant of the transfer matrix for flexural vibration in a Timoshenko Beam at high frequencies in Appendix K. To accommodate for this, non-

dimensionalized forms of transfer matrices should be used. For illustration purposes, Appendix L shows the derivation of the non-dimensionalized forms for the transfer matrices for a 3-bay planar lattice structure. Table 2 shows the calculated values of the determinant of the transfer matrix of a Timoshenko beam (using program DET2.BAS in Appendix K) based on various values of the non-dimensionalized parameter  $\Omega^2$  obtained in Appendix L using both single precision and double precision on all variables. It is observed that the value of the determinant is not calculated correctly to be unity when  $\Omega^2$  reaches a value of 4 for single precision calculations, and a value of 7.5 for double precision calculations. Furthermore, the disagreement in the calculated results for single precision calculations and double precision calculations again justifies the inaccuracies of the computed results due to operations of large numbers.

Another alternative is to use Cayley-Hamilton theorem [7] when multiplying transfer matrices. Cayley-Hamilton theorem expands the product of a number of transfer matrices as a linear combination of  $n$  independent analytical functions of the transfer matrix  $T$ , where  $n$  denotes the dimension of  $T$ . According to the theorem,

$$T^k = \sum_{j=1}^{n/2} a_j \frac{T^j + T^{-j}}{2} + b_j \frac{T^j - T^{-j}}{2} \quad (76)$$

where  $a_j$ 's and  $b_j$ 's are constants and  $K$  is the number of times transfer matrix  $T$  is multiplied to itself. The constants  $a_j$ 's and  $b_j$ 's can be obtained by substituting for  $T$  the eigenvalues of  $T$  in eqn. (72). For example, if transfer matrix  $T$  is a 4x4 matrix, and if the four eigenvalues of  $T$  (refer to Appendix K) are substituted successively to eqn. (76), four independent equations, with  $a_j$ 's and  $b_j$ 's as the only unknowns, are obtained. By solving these four simultaneous equations, the unknowns  $a_j$ 's and  $b_j$ 's can be determined. Eqn. (76) can even be applied to obtain  $T^k$ . The Cayley-Hamilton theorem becomes favorable when the number of multiplication exceeds the dimension of the transfer matrix. This is because, according to eqn. (76), the highest power in  $T$  is  $n/2$ , independent on the value of  $k$ .

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Table 1. Tabulated Values of the Natural Frequencies of a Six Segment Rod (refer to Figure 19).

Principal Modes	Natural Frequency (rad/sec) $\omega = \frac{(2n-1)\pi}{12\ell} \sqrt{\frac{E}{\rho}}$ n=1,2,...
1	5335.73
2	16007.19
3	26678.65
4	37350.11
5	48021.57
6	58693.03
7	69364.49
8	80035.95
9	90707.41

Table 2. Computed Values of The Determinant of Transfer matrix for Timoshenko Beam for Various Values of  $\omega^2$  using Double and Single Precisions.\*

$\omega^2$	Determinant of Transfer Matrix for Timoshenko Beam	
	Single precision	Double Precision
0	1.000000	1.000000
0.5	1.000000	1.000001
1.0	1.000015	1.000013
1.5	1.000191	1.000031
2.0	1.000328	0.999764
2.5	0.999418	0.999844
3.0	0.998260	1.000488
3.5	1.001465	0.999756
4.0	0.990234	1.002930
4.5	0.860351	1.003906
5.0	1.193359	0.980469
5.5	-0.359375	0.996094
6.0	1.283203	1.003320
6.5	4.146484	1.085938
7.0	-17.234380	1.041992
7.5	7.898438	1.023438
8.0	-95.906250	0.781250
8.5	287.125000	0.625000
9.0	418.250000	0.250000
9.5	360.000000	3.000000
10.0	-1380.500000	-1.000000

- 
- \* Elastic modulus is  $7.46 \times 10^{10} P_s$  ( $10.8 \times 10^6$  psi), shear modulus is  $2.75 \times 10^{10} P_s$  ( $4.0 \times 10^6$  psi), cross sectional area is  $6.04 \times 10^{-3} m^2$  ( $9.375 \times 10^{-2} in$ ), second moment of area inertia is  $4.55 \times 10^{-10} in^4$  ( $1.098 \times 10^{-3} in^4$ ), mass density is  $7.2 kg/m^3$  ( $0.1 lbf/in^3$ ), radius of gyration is  $2.71 \times 10^{-3} m$  ( $1.0838 \times 10^{-1} in$ ), and length in  $2.43 \times 10^{-1} m$  ( $9.75 in$ )

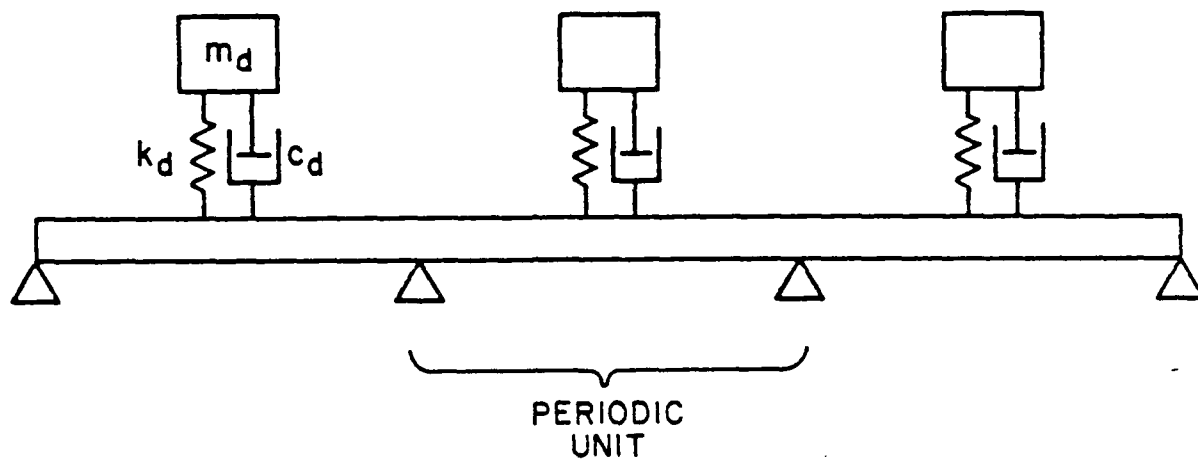


Fig. 1 A simple periodic structure.



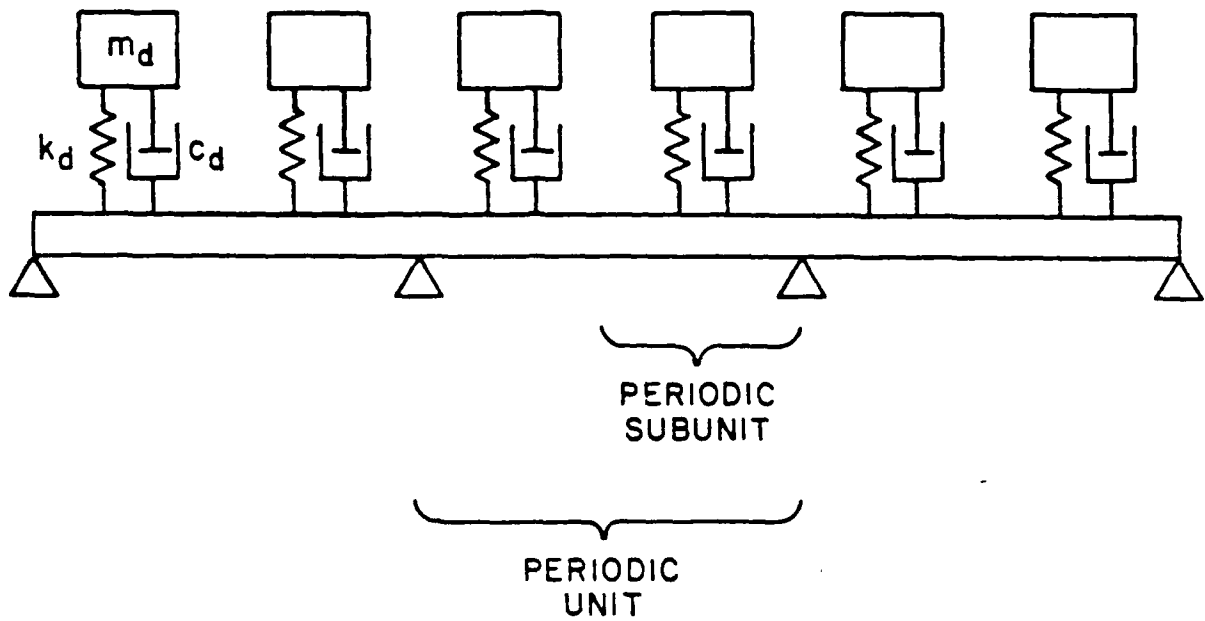


Fig. 2 A compound periodic structure.

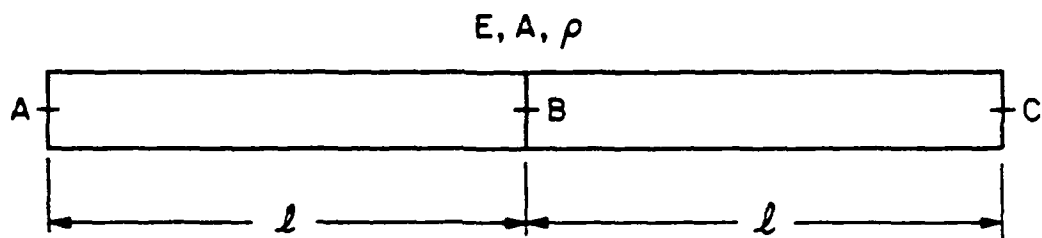


Fig. 3 An elastic rod of length  $2l$  with distributed mass.

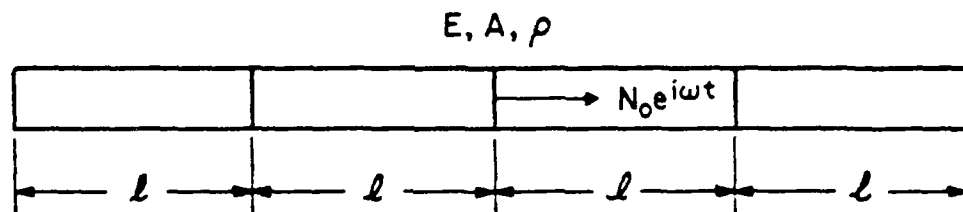


Fig. 4 An elastic rod with distributed mass and sinusoidal axial force excitation at point C.



Fig. 5 Internal forces at C. (Refer to Fig. 4)

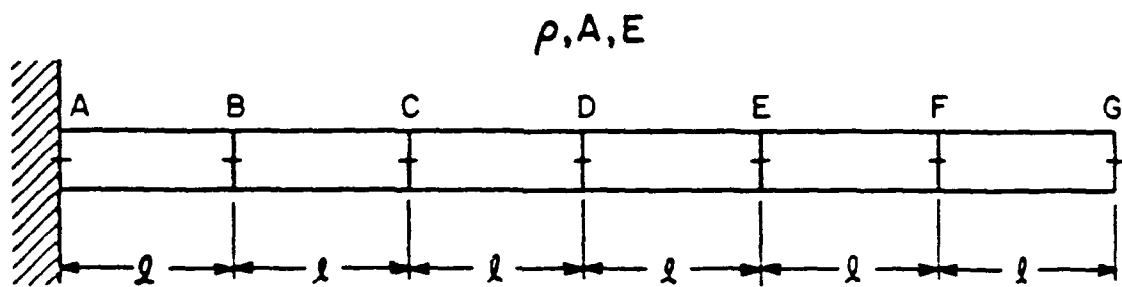


Fig. 6 An elastic rod of length  $6l$  with distributed mass.



Fig. 7 Sign convention for longitudinal vibration in an elastic rod.

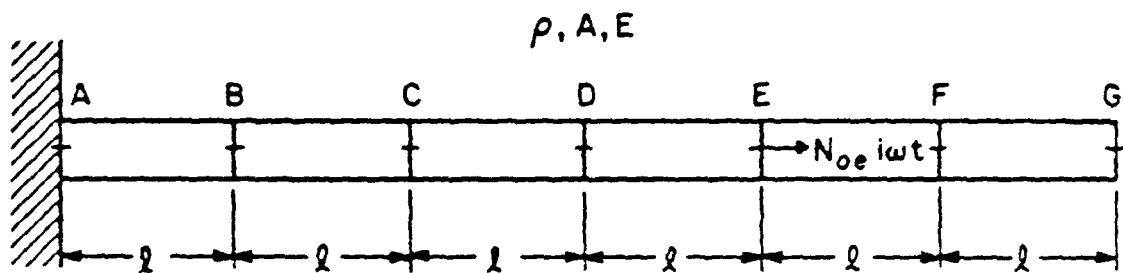


Fig. 3 An elastic rod loaded by a sinusoidal axial force at E.

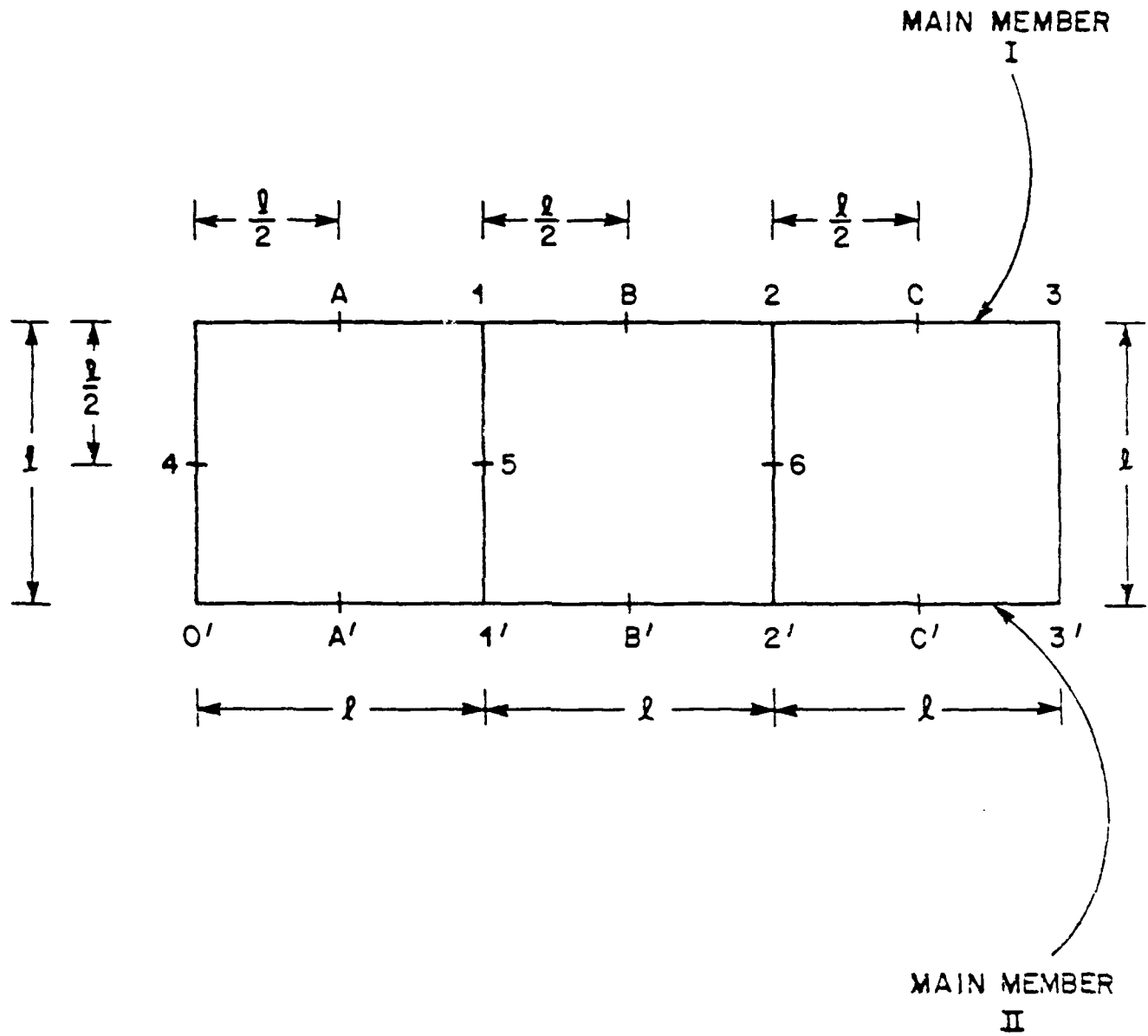


Fig. 9 A 3-bay planar lattice structure.



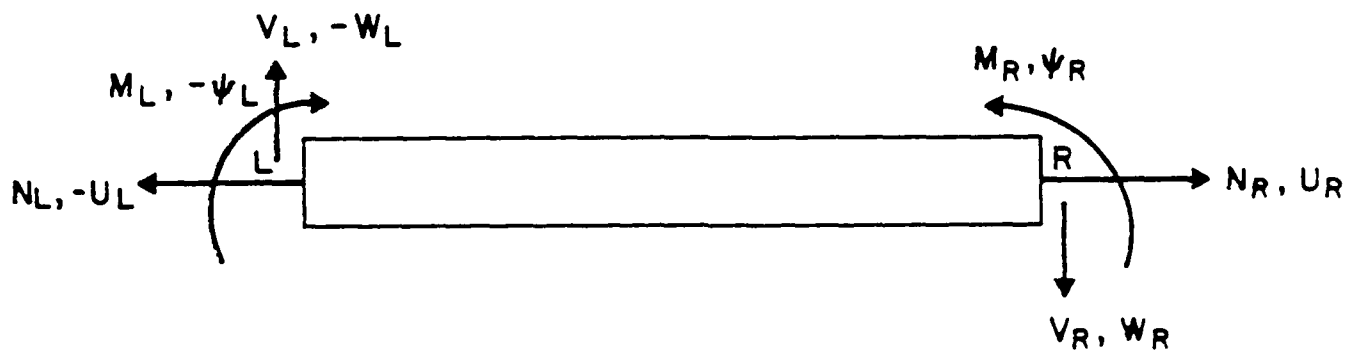


Fig.10 Sign convention for the longitudinal and flexural vibration in a Timoshenko beam.

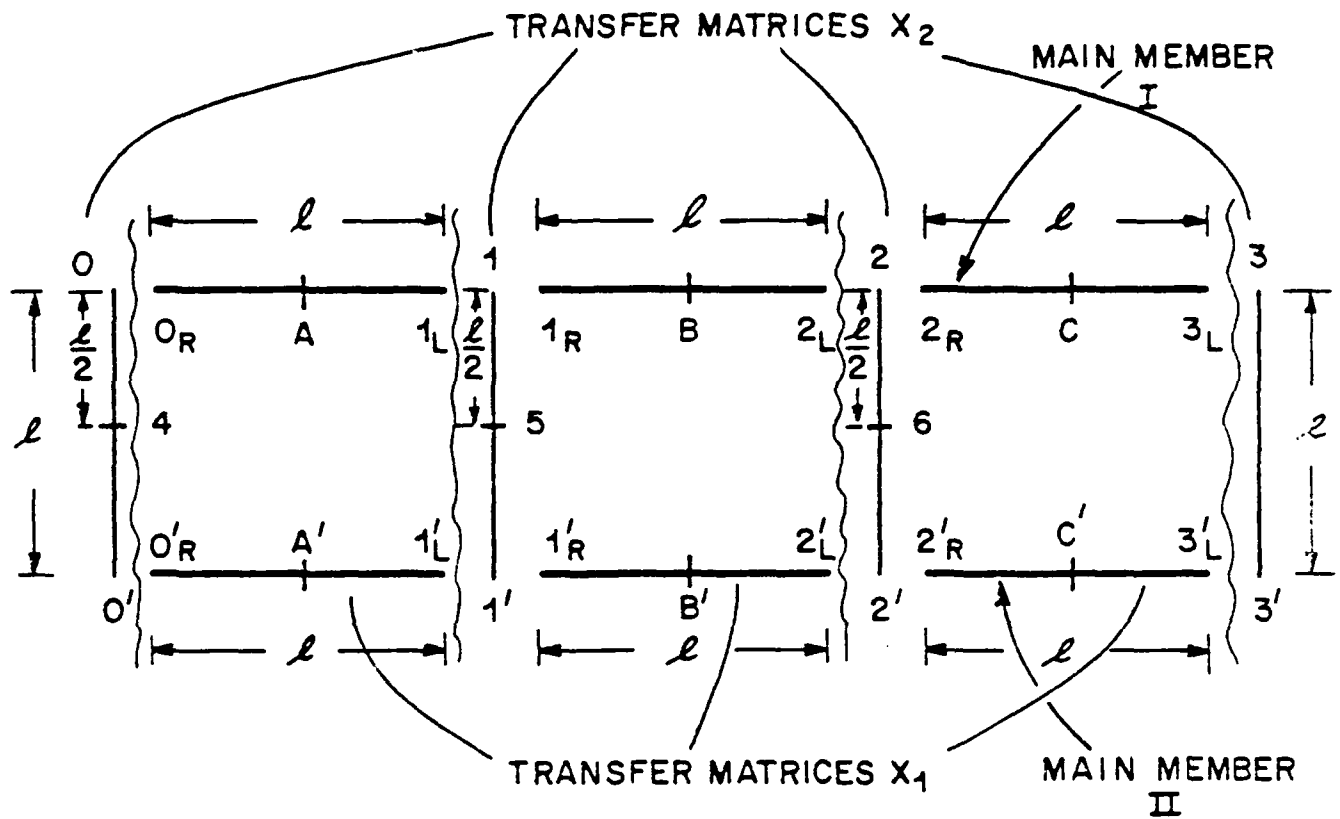


Fig. 11 A planar structure sectioned into constituent parts which make up the transfer matrices  $X_1$  and  $X_2$ .

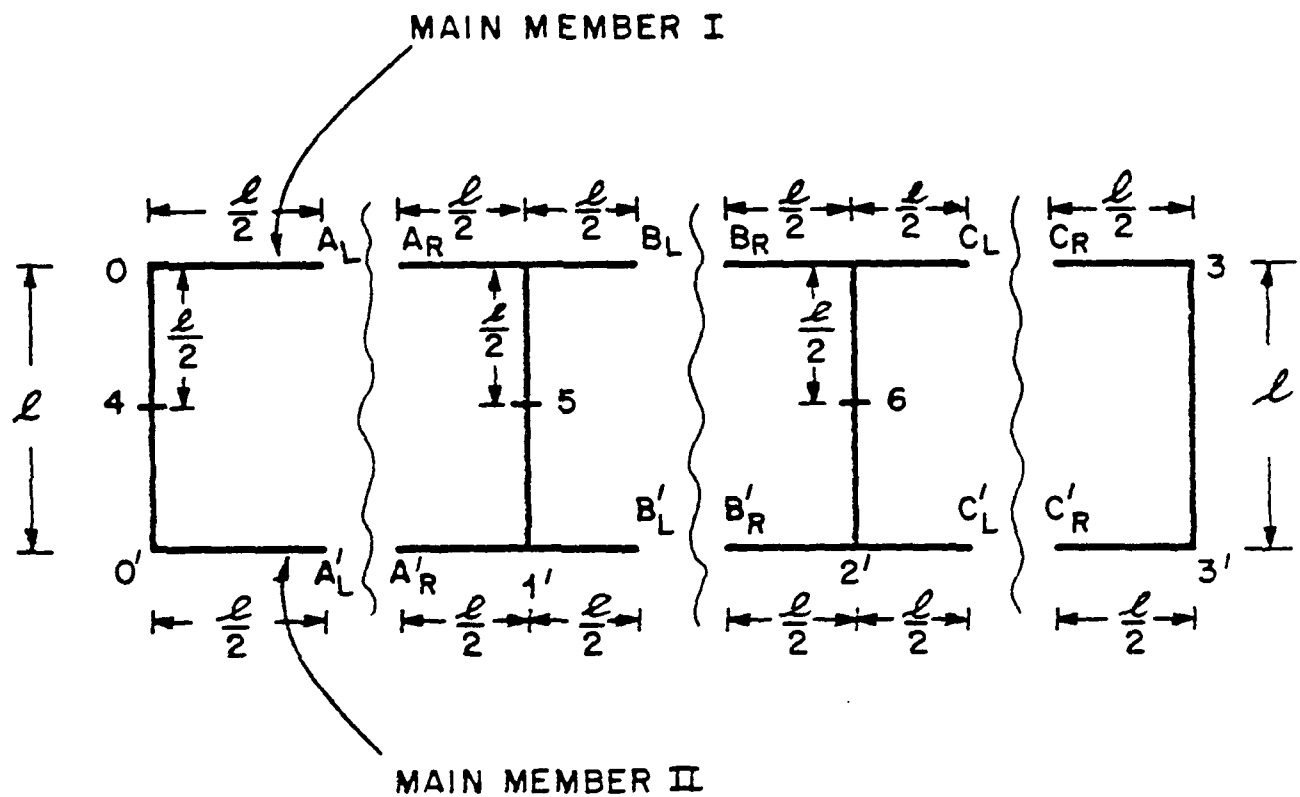


Fig. 12 A 3-bay planar structure sectioned into basic periodic units and half units.

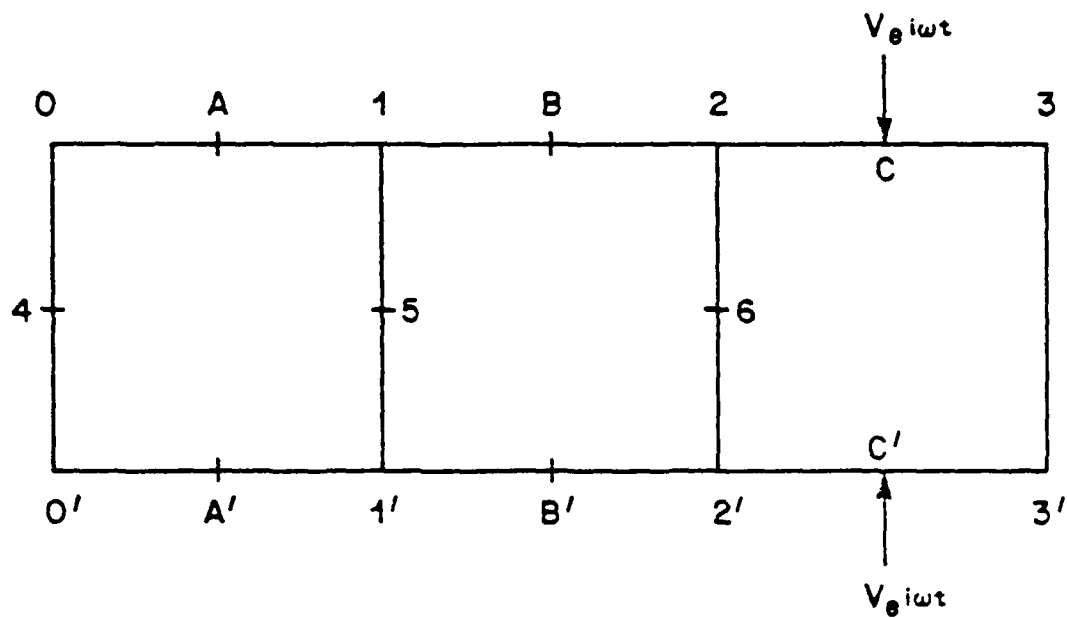


Fig. 13 The 3-bay planar structure loaded by sinusoidal shear forces.

LENGTH OF EACH BAR = 1

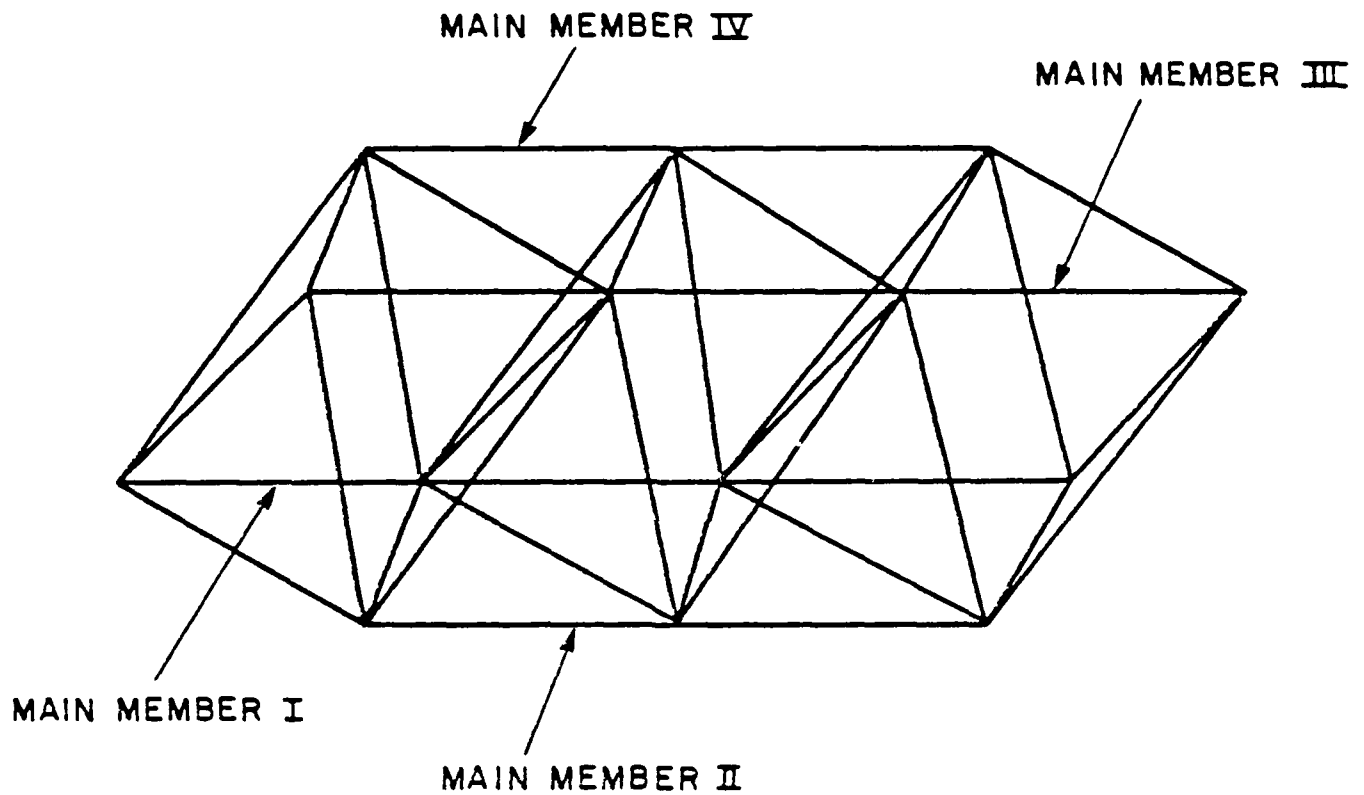


Fig. 14 A tetrahedral truss with repeating periodic units.

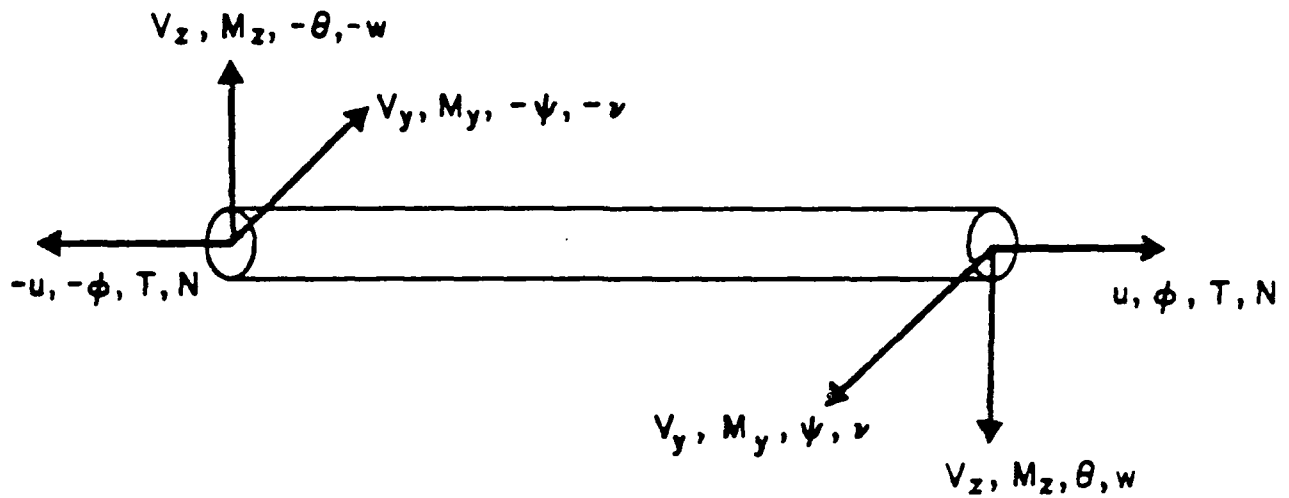


Fig. 15 Sign convention for the forces and displacements in a connecting bar of a tetrahedral truss, expressed in local coordinates.

LENGTH OF EACH BAR = 1

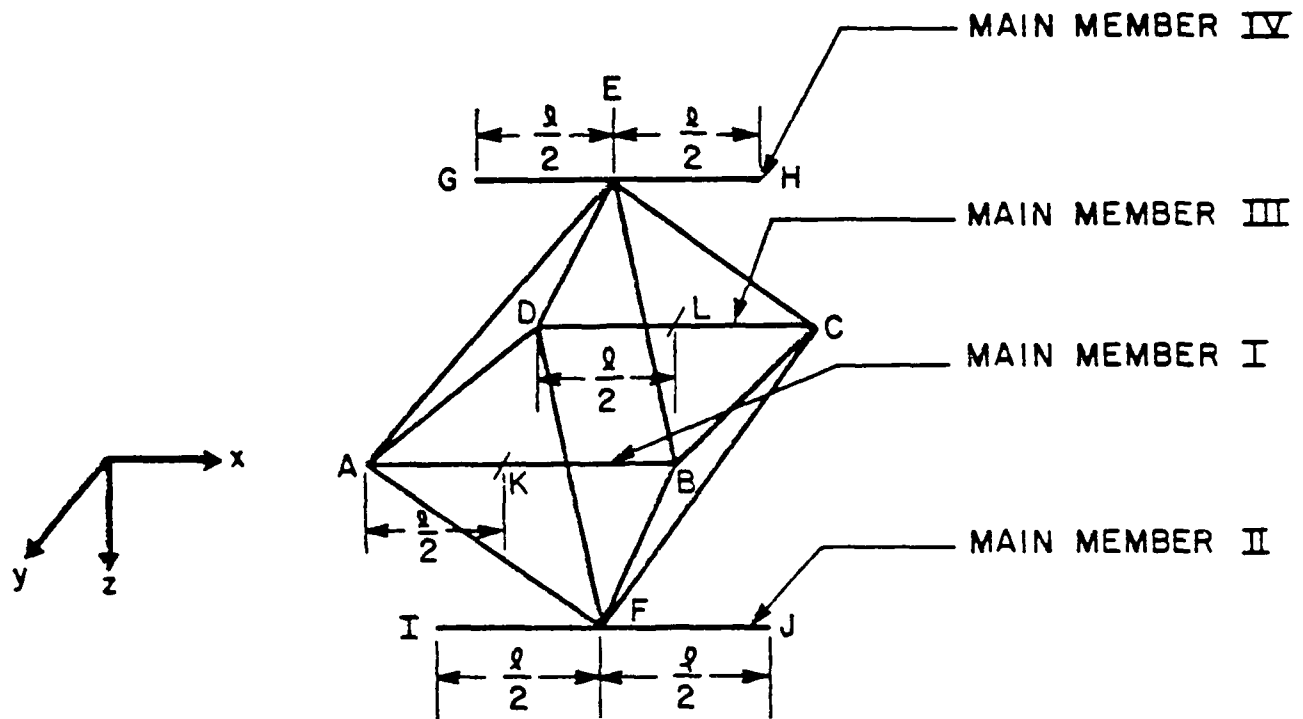


Fig. 16 A segment of the tetrahedral truss.

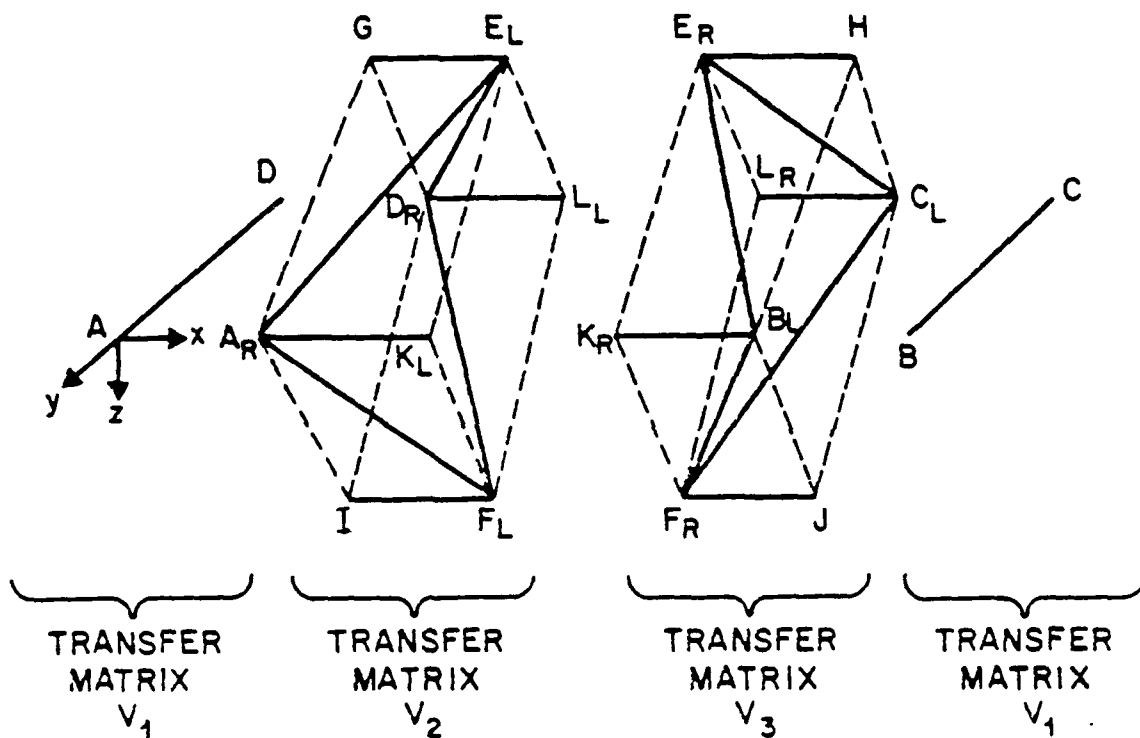


Fig. 17 Sectioning of the segment into constituent parts comprising the transfer matrices.



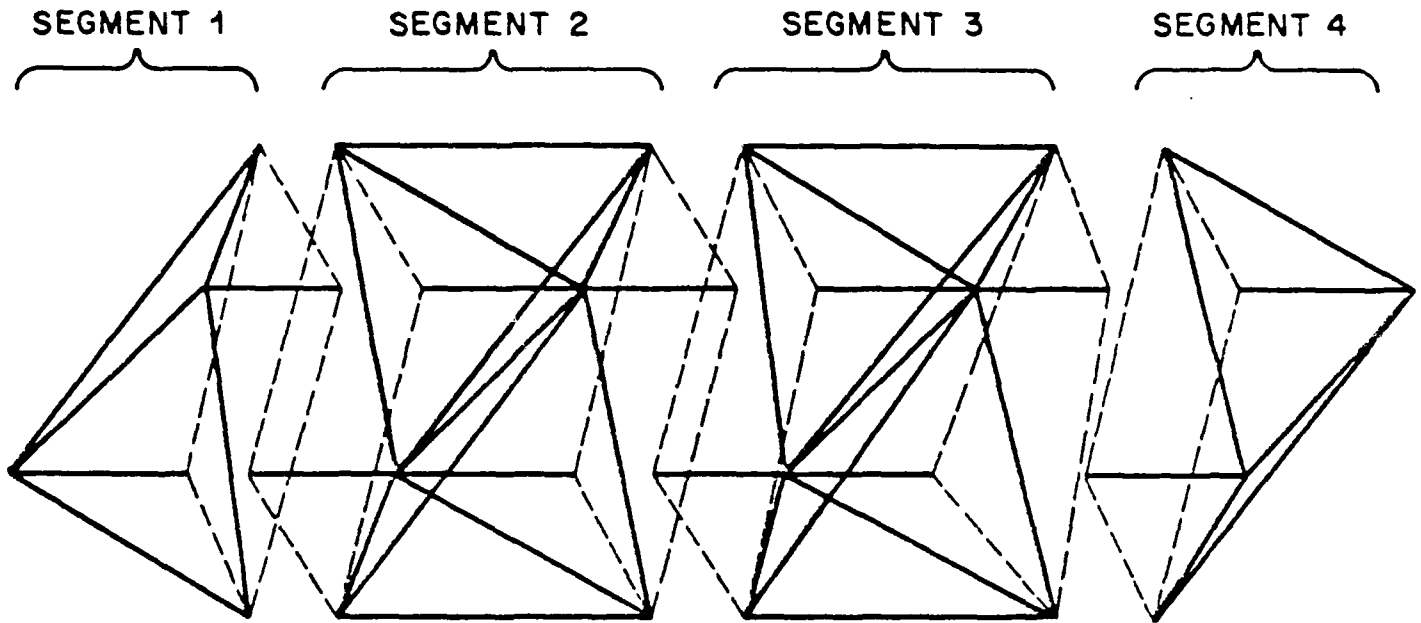


Fig. 18 The Tetrahedral truss (Fig. 14) sectioned into four basic segments.

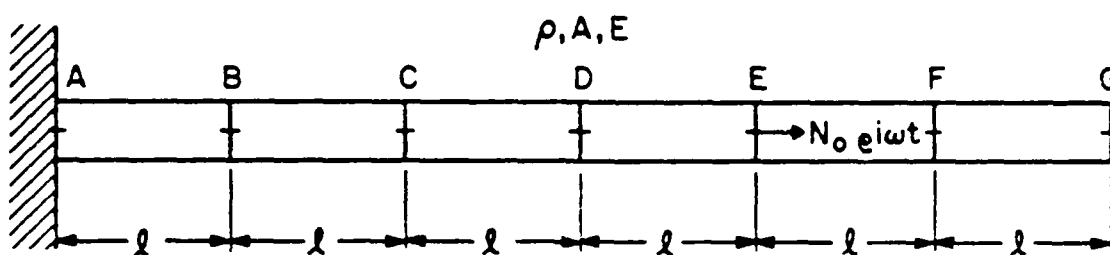


Fig. 19 An elastic rod clamped at the left end and free at the right end, loaded by external excitation at point E.

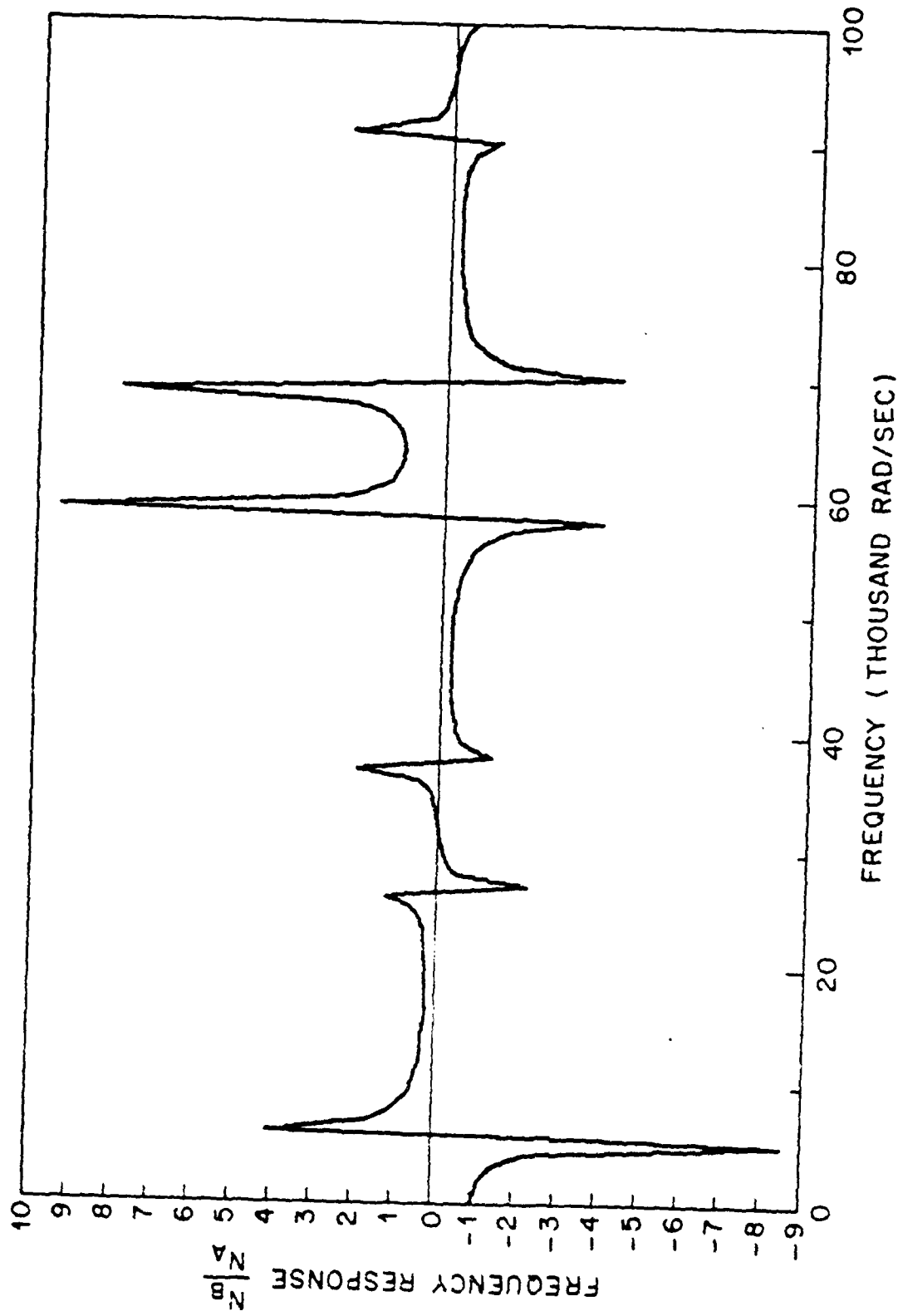


Fig. 20 Frequency force response of force at B due to unit sinusoidal axial force excitation at E (Refer to Fig. 19).

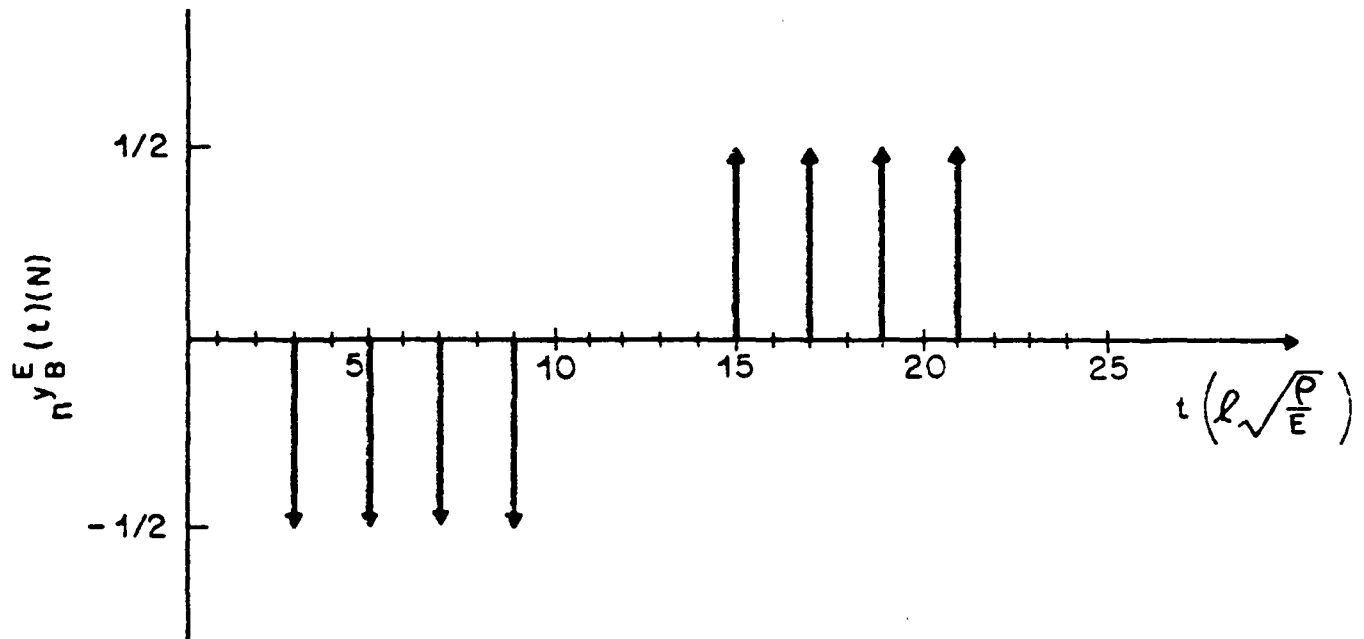


Fig. 21 Impulse response of force in rod at B due to a positive unit impulse force at E (Refer to Fig. 19).

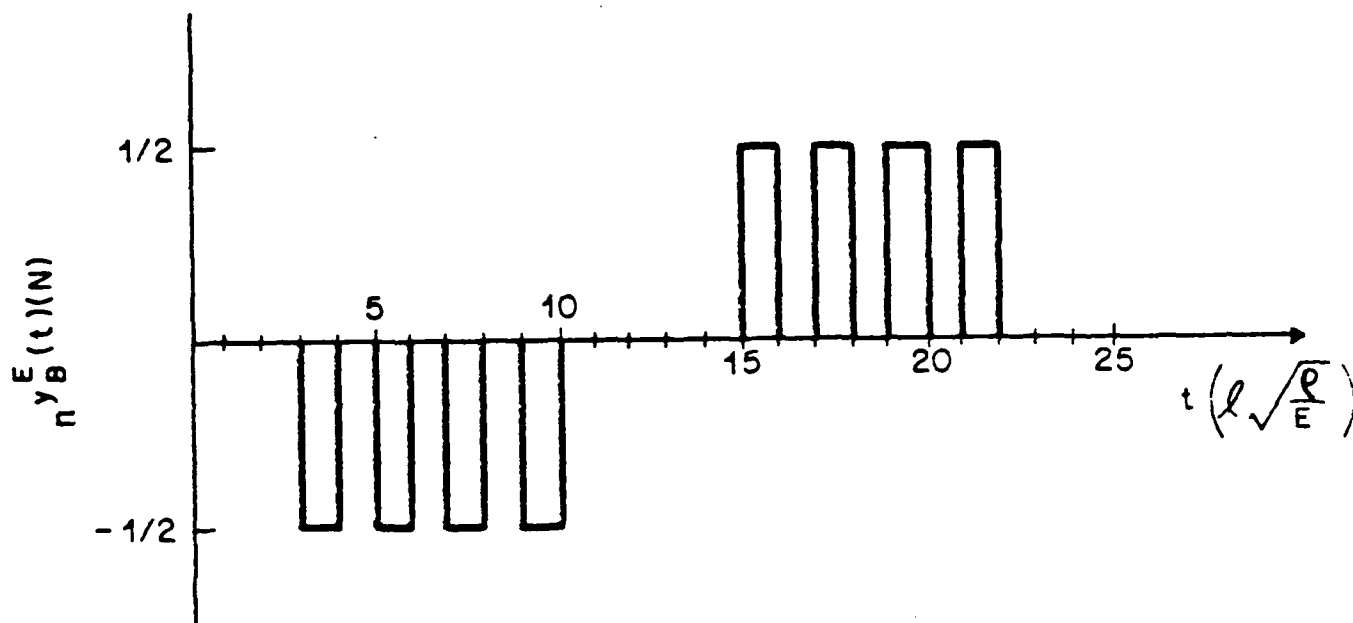


Fig. 22 Force response at B due to a positive square force pulse excitation at E (Refer to Fig. 19).

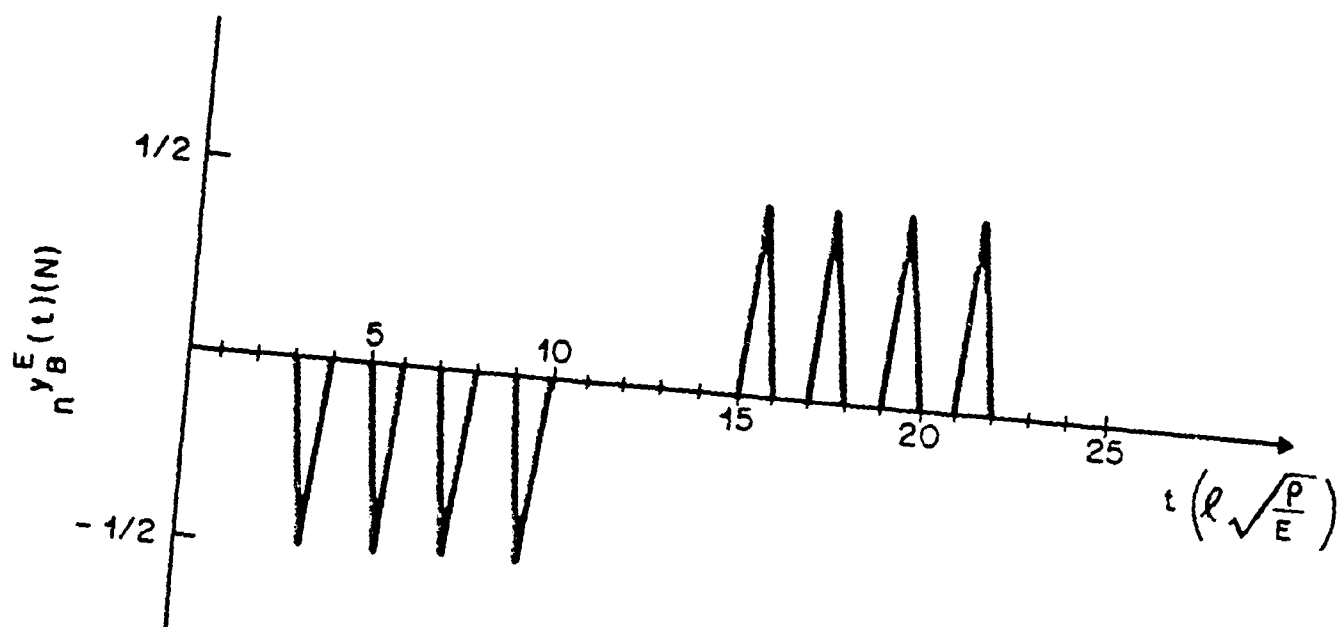


Fig. 23 Force response at B due to triangular pulse force excitation at E (Refer to Fig. 19).

## APPENDIX A

### PROPAGATION CONSTANTS AS APPLIED TO THE ANALYSIS OF WAVE PROPAGATION IN PERIODIC STRUCTURES

The characteristics of wave propagation and vibration in periodic structures are best understood in terms of propagating and non-propagating wave motions. In general, waves can propagate in some frequency bands and not in others [2]. In other words, periodic units in periodic structures behave like band-pass filters, responding very efficiently in certain frequency bands only.

Such a characteristic is commonly described by a propagation constant  $\mu$  [1,2] which is described by the nature of the periodic unit and the corresponding excitation frequency. The harmonic motion at one end of a periodic unit is equal to  $e^{-\mu}$  times the motion at the other end from which the wave is travelling. A propagation constant  $\mu$  can be real, imaginary or complex, and its value always occurs in positive and negative pairs, which correspond to identical but opposite going waves. The real part of  $\mu$  is called the attenuation  $\alpha$ , and the imaginary part is called the phase constant (or wave number)  $\kappa$ . Purely imaginary propagation constants are known to be associated with waves which propagate energy, whereas purely real propagation constants belong to waves of no energy flow [1]. The frequency bands of the real part of  $\mu$  are called the propagation zones; other frequency bands are called the attenuation zones. The number of possible propagation constants (and the corresponding waves), for a periodic unit at a particular frequency, is equal to twice the number of state vectors (or coupling coordinates) between adjacent periodic units [1]. For example, for the flexural vibrations of a beam, there are eight propagation constants corresponding to the four coupling coordinates which are, namely, the transverse displacement, the angle of rotation of the cross section, the shear force and the moment. For a particular value of  $\mu$ , the positive-going waves, each as a function of  $x$ , and of the form [2]

$$W_{+}(x) = \sum_{n=-\infty}^{\infty} A_n e^{i[\omega - (\mu + 2n\pi)x/\ell]} \quad (A1)$$

and the negative-going waves, each as a function of  $x$ , are of the form [2]

$$W_{-}(x) = \sum_{n=-\infty}^{\infty} B_n e^{i[\omega - (\mu + 2n\pi)x/\ell]} \quad (A2)$$

Where  $A_n$  and  $B_n$  are constants and  $W$  represents the wave parameter of interest (for example, stress wave in a periodic structure).



## APPENDIX B

### TRANSFER MATRICES FOR LONGITUDINAL VIBRATION IN AN ELASTIC ROD WITH DISTRIBUTED MASS

In this appendix, the transfer matrices (from left to right and from right to left) for longitudinal vibration in an elastic rod with distributed mass are derived using the classical wave equation for longitudinal vibration in an elastic rod.

Fig. B1 shows an elastic rod with distributed mass together with the sign convention adopted for the forces and the displacements. The rod has modulus of elasticity  $E$ , cross-sectional area  $A$ , mass density  $\rho$  and length  $\ell$ . Furthermore,  $N(x, t)$  represents the internal axial force and  $u(x, t)$  represents the longitudinal displacement in the rod.

Consider a small element of the rod as shown in Fig. B2. Using the momentum principle,

$$\frac{\partial N}{\partial X} = \rho A \frac{\partial^2 u(x, t)}{\partial t^2} \quad (B1)$$

By the definitions of stress and strain,

$$N = \sigma A ,$$

$$\epsilon = \frac{\partial u(x, t)}{\partial X} ,$$

$$\text{and } \sigma = E \epsilon ,$$

which give

$$\frac{\partial N}{\partial x} = AE \frac{\partial^2 u(x, t)}{\partial x^2} \quad (B2)$$

Equating eqns. (B1) and (B2), the wave equation is given as

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 u(x,t)}{\partial x^2} \quad (B3)$$

Now let  $u(x,t) = u(x)\sin(\omega t + \phi)$  where  $\omega$  is the circular frequency of vibration and  $\phi$  is the displacement. Substituting this assumed form into eqn. (B3) gives

$$-\omega^2 \sin(\omega t + \phi) \frac{d^2 u(x)}{dx^2} - \frac{\omega^4 \rho}{E} \sin(\omega t + \phi) u(x) \quad (B4)$$

Cancelling the terms  $-\omega^2 \sin(\omega t + \phi)$  from eqn. (B4) gives

$$\frac{d^2 u(x)}{dx^2} + \frac{\omega^2 \rho}{E} u(x) = 0 \quad (B5)$$

Eqn. (B5) has a solution of the form

$$u(x) = C_1 \sin(\theta \frac{x}{\ell}) + C_2 \cos(\theta \frac{x}{\ell}) \quad (B6)$$

where  $\theta = \ell \omega \sqrt{\frac{\rho}{E}}$  and where  $C_1$  and  $C_2$  are constants.

#### TRANSFER MATRIX FROM LEFT TO RIGHT

To obtain the transfer matrix from left to right for a rod, adopt the left end of the rod in Fig. B1 as the origin for the  $x$  axis. Applying the boundary conditions to eqn. (B6),

$$u = u_L, \text{ at } x = 0$$

$$u = u_R, \text{ at } x = \ell$$

where  $u_L$  and  $u_R$  are known quantities and solving for the constants in eqn. (B6),

$$u(x) = \frac{u_R - u_L \cos \theta}{\sin \theta} \sin(\theta \frac{x}{\ell}) + u_L \cos(\theta \frac{x}{\ell}) \quad (B7)$$

Using eqn. (B2) or  $N = AE \frac{du}{dx}$ , and calling the resulting boundary values as,

$$N = N_L, \text{ at } x = 0$$

$$\text{and } N = N_R, \text{ at } x = \ell,$$

eqn. (B7) gives

$$u_R = \cos \theta u_L + \frac{\ell}{EA} \frac{\sin \theta}{\theta} N_L \quad (B8)$$

$$N_R = -\frac{\theta EA}{\ell} \sin \theta u_L + \cos \theta N_L \quad (B9)$$

Writing eqns. (B8) and (B9) in matrix form,

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_R = \begin{bmatrix} \cos \theta & \frac{\ell}{EA} \frac{\sin \theta}{\theta} \\ -\mu \ell \omega^2 \frac{\sin \theta}{\theta} & \cos \theta \end{bmatrix} \begin{Bmatrix} u \\ N \end{Bmatrix}_L \quad (B10)$$

where  $\mu = \rho A$  and the subscripts  $R$  and  $L$  denote the right and left ends of the bar, respectively.

Thus, from eqn. (B10), the transfer matrix  $T$  from left to right is:

$$T = \begin{bmatrix} \cos \theta & \frac{\ell}{EA} \frac{\sin \theta}{\theta} \\ -\mu \ell \omega^2 \frac{\sin \theta}{\theta} & \cos \theta \end{bmatrix} \quad (B11)$$

#### TRANSFER MATRIX FROM RIGHT TO LEFT

To obtain the transfer matrix from right to left for a rod, adopt the right end of the rod in Fig. B1 as the origin. Then, applying the boundary conditions to eqn. (B6),

$$u = u_R, \text{ at } x = 0$$

$$\text{and } u = u_L, \text{ at } x = -\ell,$$

where  $u_R$  and  $u_L$  are known quantities and solving for the constants in eqn. (B6),

$$u(x) = \frac{u_R \cos \theta - u_L}{\sin \theta} \sin\left(\theta \frac{x}{\ell}\right) + u_R \cos\left(\theta \frac{x}{\ell}\right) \quad (\text{B12})$$

Using eqn. (B2) and calling the boundary values as,

$$N = N_R, \text{ at } x = 0$$

$$\text{and } N = N_L, \text{ at } x = -\ell,$$

eqn. (B12) gives

$$u_L = \cos \theta u_R - \frac{\ell}{EA \theta} \sin \theta N_R \quad (\text{B13})$$

$$N_L = \frac{EA \theta}{\ell} \sin \theta u_R + \cos \theta N_R \quad (\text{B14})$$

Writing eqns. (B13) and (B14) in matrix form,

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_L = \begin{bmatrix} \cos \theta & -\frac{\ell}{EA} \frac{\sin \theta}{\theta} \\ \mu \ell \omega^2 \frac{\sin \theta}{\theta} & \cos \theta \end{bmatrix} \begin{Bmatrix} u \\ N \end{Bmatrix}_R \quad (\text{B15})$$

From eqn. (B15), the transfer matrix  $T^{-1}$  from right to left is

$$T^{-1} = \begin{bmatrix} \cos \theta & -\frac{\ell}{EA} \frac{\sin \theta}{\theta} \\ \mu \ell \omega^2 \frac{\sin \theta}{\theta} & \cos \theta \end{bmatrix} \quad (\text{B16})$$

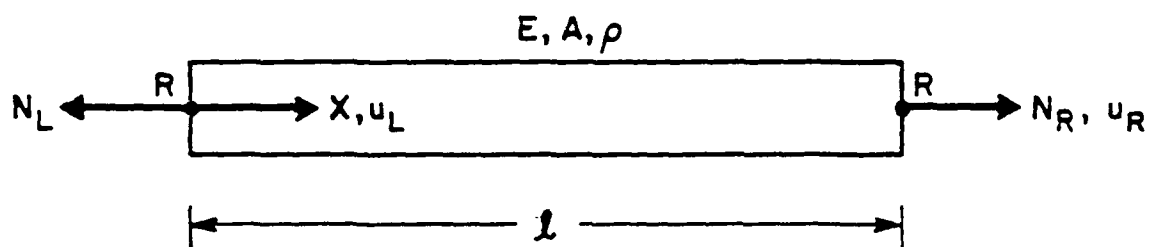


Fig. B1 An elastic rod with distributed mass.

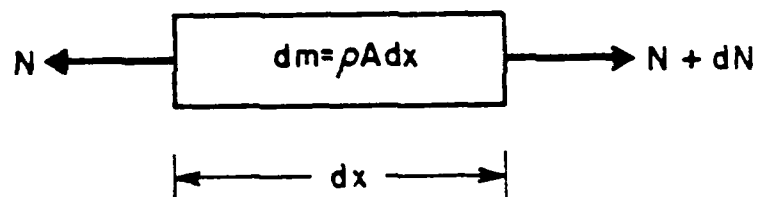


Fig. B2 Free-body diagram of an element of the rod.

## APPENDIX C

### TRANSFER MATRIX FOR LONGITUDINAL VIBRATION IN AN ELASTIC ROD WITH DISTRIBUTED MASS AND DAMPING

In this appendix, the transfer matrix from left to right for longitudinal vibration in an elastic rod with distributed mass and damping is derived using the wave equation. The derivation of the transfer matrix from right to left in an elastic rod with distributed mass and damping follows a similar approach and therefore is not discussed.

Fig. C1 shows an elastic rod with distributed mass and damping together with the sign convention adopted for the forces and the displacements. The rod has modulus of elasticity  $E$ , material damping  $c$ , mass density  $\rho$ , cross-sectional area  $A$  and length  $\ell$ . Furthermore,  $N(x, t)$  represents the internal axial force in the rod and  $u(x, t)$  represents the longitudinal displacement in the rod.

Consider a small element of the rod as shown in Fig. C2. Using the momentum principle,

$$\frac{\partial N}{\partial X} = \rho A \frac{\partial^2 u(x, t)}{\partial t^2} \quad (C1)$$

Now assume that the rod can be modeled as a simple Voigt material [11] with elastic modulus  $E$  and damping constant  $c$  such that the stress-strain relation is given by

$$\sigma = E \epsilon + c \frac{d \epsilon}{dt} \quad (C2)$$

By the definitions of stress and strain,

since  $N = \sigma A$

$$\text{and } \epsilon = \frac{\partial u(x,t)}{\partial x},$$

$$\frac{\partial N}{\partial x} = A \left[ E \frac{\partial^2 u(x,t)}{\partial x^2} + c \frac{\partial}{\partial t} \left( \frac{\partial^2 u(x,t)}{\partial x^2} \right) \right] \quad (C3)$$

Equating eqns. (C1) and (C3) and rearranging,

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{1}{\rho} \left[ E \frac{\partial^2 u(x,t)}{\partial x^2} + c \frac{\partial}{\partial t} \left( \frac{\partial^2 u(x,t)}{\partial x^2} \right) \right] \quad (C4)$$

Now assume that

$$u(x,t) = u(x)e^{i\omega t} \quad (C5)$$

After substituting eqn. (C5) into eqn. (C4) and cancelling the  $e^{i\omega t}$  terms,

$$\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\omega^2 \rho}{(E + ic)} u(x) = 0 \quad (C6)$$

Eqn. (C6) is the wave equation for longitudinal vibration in an elastic rod with distributed mass and damping. It has a solution of the form.



$$u(x) = C_1 \sin \left( \theta \frac{x}{\ell} \right) + C_2 \cos \left( \theta \frac{x}{\ell} \right) \quad (C7)$$

$$\text{where } \theta = \ell \omega \sqrt{\frac{\rho}{E + ic}}$$

and there  $C_1$  and  $C_2$  are constants.

Applying the boundary conditions,

$$u = u_L, \text{ at } x = 0$$

$$\text{and } u = u_R, \text{ at } x = \ell$$

where  $u_L$  and  $u_R$  are known quantities and calling the axial forces at the left end and the right end of the rod  $N_L$  and  $N_R$ , respectively, from eqn. (C7),

$$u_R = \cos \theta u_L + \frac{\ell}{EA} \frac{\sin \theta}{\theta} N_L \quad (C8)$$

$$\text{and } N_R = -\frac{EA \theta}{\ell} \sin \theta u_L + \cos \theta N_L \quad (C9)$$

Writing eqns. (C8) and (C9) in matrix form,

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_R = \begin{bmatrix} \cos \theta & \frac{\ell}{EA} \frac{\sin \theta}{\theta} \\ -\mu \ell \omega^2 \frac{\sin \theta}{\theta} & \cos \theta \end{bmatrix} \begin{Bmatrix} u \\ N \end{Bmatrix}_L \quad (C10)$$

where  $\mu = \rho A$ .

The transfer matrix  $T$  required is thus

$$T = \begin{bmatrix} \cos \theta & \frac{\ell}{EA} \frac{\sin \theta}{\theta} \\ -\mu \ell \omega^2 \frac{\sin \theta}{\theta} & \cos \theta \end{bmatrix} \quad (C11)$$

The relationship between the damping constant  $c$  and the attenuation parameter  $\alpha$  has been investigated in [12] and is given by

$$c = \frac{E}{\omega} \tan \left( 2 \tan^{-1} \frac{\alpha c_p}{\omega} \right) \quad (C12)$$

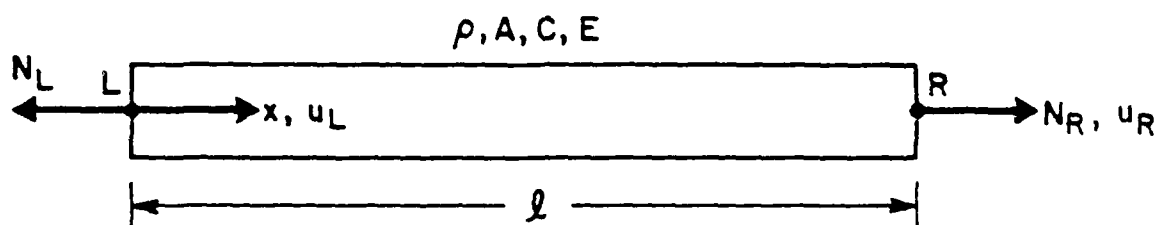


Fig. C1 An elastic rod with distributed mass and damping.

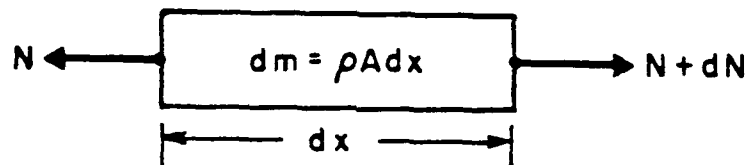


Fig. C2 Free-body diagram of an element of the rod.

APPENDIX D

TRANSFER MATRIX FOR FLEXURAL VIBRATION IN AN  
ELASTIC BAR INCLUDING THE EFFECT OF SHEAR DEFLECTION  
AND ROTARY INERTIA

The transfer matrix for flexural vibration in an elastic bar including the effect of shear deflection and rotary inertia is derived in this appendix. Fig. D1 shows an elastic bar with distributed mass and the sign convention chosen. The bar has modulus of elasticity  $E$ , shear modulus  $G$ , second moment of area inertia about the  $y$  axis  $J$ , mass per unit length  $\mu$ , and radius of gyration  $r$ . In addition,  $w$  denotes the lateral displacement,  $\phi$  denotes the rotation of the cross-sectional area about the  $y$  axis and  $M$  and  $V$  represent the moment and the shear force, respectively.

Consider first an element of the bar as shown in Fig. D2, which gives

$$V = GA_s \left( \frac{dw}{dx} + \psi \right) \quad (D1)$$

where  $GA_s = GA/K_s$ , is the shear stiffness and  $K_s$  is the form factor which depends on the shape of the cross-sectional area.

The constitutive bending relation for a bar is

$$M = EJ \frac{d\phi}{dx} \quad (D2)$$

Now consider Fig. D3, which shows a free-body diagram of an element of the bar. Equilibrium consideration give the following equations:

$$\frac{dM}{dx} = V - \mu \ell^2 \omega^2 \psi \quad (D3)$$

$$\frac{dV}{dx} = -\mu \omega^2 w \quad (D4)$$

Differentiating eqn. (D1) with respect to  $x$ , using the relationship in eqn. (D2) and substituting into eqn. (D4) gives

$$\frac{d^2 w}{dx^2} + \frac{\mu \omega^2}{GA_s} w + \frac{M}{EJ} = 0 \quad (D5)$$

Now differentiating eqn. (D3) with respect to  $x$ , using the relationship from eqn. (D2) and then substituting into eqn. (D4),

$$\frac{d^2 M}{dx^2} + \frac{\mu \ell^2 \omega^2}{EJ} M + \mu \omega^2 w = 0 \quad (D6)$$

Eliminating  $M$  in eqns. (D5) and (D6)

$$\frac{d^4 w}{dx^2} + \frac{\sigma + \tau}{\ell^2} \frac{d^2 w}{dx^2} - \frac{\beta^4 - \sigma \tau}{\ell^4} w = 0 \quad (D7)$$

where  $\sigma = \frac{\mu \omega^2 \ell^2}{GA_s}$

$$\tau = \frac{\mu \ell^2 \omega^2 \ell^2}{EJ}$$

and  $\beta^4 = \frac{\mu \omega^2 \ell^4}{EJ}$

Now assume a displacement  $w$  such that

$$w = C e^{\lambda x / \ell}$$

where  $C$  is a constant. Substituting the assumed form for displacement into eqn. (E7), cancelling the term  $-C e^{\lambda x / \ell}$  from the equation, the characteristic equation is obtained as

$$\lambda^4 + (\sigma + \tau)\lambda^2 - (\beta^4 - \sigma\tau) = 0 \quad (D8)$$

The roots of eqn. (D8) are  $\pm \lambda_1$  and  $\pm j\lambda_2$  where

$$\lambda_2^2 = \sqrt{\beta^4 + \frac{1}{4}(\sigma + \tau)} \mp \frac{1}{2}(\sigma - \tau)$$

$$\text{with } \lambda_2^2 - \lambda_1^2 = \sigma + \tau$$

$$\lambda_1^2 \lambda_2^2 = \beta^4 - \sigma\tau$$

Therefore, the solution is

$$w = \bar{C}_1 e^{\lambda_1 x / \ell} + \bar{C}_2 e^{-\lambda_1 x / \ell} + \bar{C}_3 e^{j\lambda_2 x / \ell} + \bar{C}_4 e^{-j\lambda_2 x / \ell} \quad (D9)$$

where  $\bar{C}_1, \bar{C}_2, \bar{C}_3$  and  $\bar{C}_4$  are constants. Eqn. (D9) can be written in the form

$$\begin{aligned} w = & C_1 \cosh(\lambda_1 \frac{x}{\ell}) + C_2 \sinh(\lambda_1 \frac{x}{\ell}) \\ & + C_3 \cos(\lambda_2 \frac{x}{\ell}) + C_4 \sin(\lambda_2 \frac{x}{\ell}) \end{aligned} \quad (D10)$$

$$\text{where } C_1 = \bar{C}_1 + \bar{C}_2$$

$$C_2 = \bar{C}_1 - \bar{C}_2$$

$$C_3 = \bar{C}_3 + \bar{C}_4$$

$$C_4 = j(\bar{C}_3 - \bar{C}_4)$$

Examination of eqns. (D1) and (D10) shows that  $V$  and  $w$  are of the same form.

Therefore, let

$$V = A_1 \cosh(\lambda_1 \frac{x}{\ell}) + A_2 \sinh(\lambda_1 \frac{x}{\ell}) + A_3 \cos(\lambda_2 \frac{x}{\ell}) + A_4 \sin(\lambda_2 \frac{x}{\ell}) \quad (D11)$$

Using eqns. (D1), (D10) and (D11),

$$\begin{aligned} \psi = \frac{\ell^2}{\beta^4 EJ} \left\{ (\sigma + \lambda_1^2) [A_1 \cosh(\lambda_1 \frac{x}{\ell}) + A_2 \sinh(\lambda_1 \frac{x}{\ell})] \right. \\ \left. + (\sigma - \lambda_2^2) [A_3 \cos(\lambda_2 \frac{x}{\ell}) + A_4 \sin(\lambda_2 \frac{x}{\ell})] \right\} \quad (D12) \end{aligned}$$

Using eqns. (D2), (D10) and (D11),

$$\begin{aligned} M = \frac{\ell^2}{\beta^4} \left\{ (\sigma + \lambda_1^2) \frac{\lambda_1}{\ell} [A_1 \sinh(\lambda_1 \frac{x}{\ell}) + A_2 \cosh(\lambda_1 \frac{x}{\ell})] \right. \\ \left. - (\sigma - \lambda_2^2) \frac{\lambda_2}{\ell} [A_3 \sin(\lambda_2 \frac{x}{\ell}) + A_4 \cos(\lambda_2 \frac{x}{\ell})] \right\} \quad (D13) \end{aligned}$$

Writing eqns. (D10) to (D13) in matrix form,

$$\begin{Bmatrix} -w \\ \psi \\ M \\ V \end{Bmatrix} = \begin{bmatrix} \frac{\ell^3}{\beta^4 EJ} \sinh(\lambda_1 \frac{x}{\ell}) & \frac{\ell^3}{\beta^4 EJ} \cosh(\lambda_1 \frac{x}{\ell}) \\ \frac{\ell^2(\sigma + \lambda_1^2)}{\beta^4 EJ} \cosh(\lambda_1 \frac{x}{\ell}) & \frac{\ell^2(\sigma + \lambda_1^2)}{\beta^4 EJ} \sinh(\lambda_1 \frac{x}{\ell}) \\ \frac{\ell \lambda_1 (\sigma + \lambda_1^2)}{\beta^4} \sinh(\lambda_1 \frac{x}{\ell}) & \frac{\ell \lambda_1 (\sigma + \lambda_1^2)}{\beta^4} \cosh(\lambda_1 \frac{x}{\ell}) \\ \cosh(\lambda_1 \frac{x}{\ell}) & \sinh(\lambda_1 \frac{x}{\ell}) \end{bmatrix}$$



$$\left[ \begin{array}{cc} \frac{\ell^3}{\beta^4 EJ} \sin(\lambda_2 \frac{x}{\ell}) & \frac{\ell^3 \lambda_2}{\beta^4 EJ} \cos(\lambda_2 \frac{x}{\ell}) \\ \frac{\ell^2 (\sigma - \lambda_2^2)}{\beta^4 EJ} \cos(\lambda_2 \frac{x}{\ell}) & \frac{\ell^2 (\sigma - \lambda_2^2)}{\beta^4 EJ} \sin(\lambda_2 \frac{x}{\ell}) \\ \frac{\ell^2 \lambda_2 (\sigma - \lambda_2^2)}{\beta^4} \sin(\lambda_2 \frac{x}{\ell}) & \frac{\ell \lambda_2 (\sigma - \lambda_2^2)}{\beta^4} \cos(\lambda_2 \frac{x}{\ell}) \\ \cos(\lambda_2 \frac{x}{\ell}) & \sin(\lambda_2 \frac{x}{\ell}) \end{array} \right] \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} \quad (D14)$$

Imposing the boundary values,

$$w = w_L ; \psi = \psi_L , \text{ at } x = 0$$

$$w = w_R ; \psi = \psi_R , \text{ at } x = \ell ,$$

where  $w_L, w_R, \psi_L, \psi_R$ , are known quantities. Calling the shear forces and moments at the right end and the left end of the beam as  $V_R$  and  $M_R, V_L$  and  $M_L$ , respectively, eliminating the constants  $A_1, A_2, A_3$ , and  $A_4$ , the transfer matrix is obtained which relates the state vectors at  $x = 0$  and  $x = \ell$  as

$$\begin{Bmatrix} -w \\ \psi \\ M \\ V \end{Bmatrix}_R = \begin{bmatrix} c_0 - \alpha_2 & \ell[c_1 - (\sigma - \tau)c_3] \\ \frac{\beta^4}{\ell} c_3 & c_0 - \tau c_2 \\ \frac{\beta^4 EJ}{\ell^2} c_3 & \frac{EJ}{\ell} [-\tau c_1 + (\beta^4 + \tau^2)c_3] \\ \frac{\beta^4 EJ}{\ell^3} (c_1 - \alpha_3) & \frac{\beta^4 EJ}{\ell^2} c_2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\ell^2 c_2}{EJ} & \frac{\ell^3}{\beta^4 EJ} [-c_1 + (\beta^4 + \sigma^2) c_3] \\ \frac{\ell(c_1 - \tau c_3)}{EJ} & \frac{\ell^2}{EJ} c_2 \\ c_0 - \tau c_2 & \ell[c_1 - (\sigma + \tau) c_3] \\ \frac{\beta^4}{\ell} c_3 & c_0 - \alpha_2 \end{bmatrix} \begin{Bmatrix} -w \\ \psi \\ M \\ V \end{Bmatrix}_L$$

(D15)

$$\text{with } \Lambda = \frac{1}{\lambda_1^2 + \lambda_2^2}$$

$$\begin{aligned} c_0 &= \Lambda(\lambda_2^2 \cosh \lambda_1 + \lambda_1^2 \cos \lambda_2) \\ c_1 &= \Lambda\left(\frac{\lambda_2^2}{\lambda_1} \cosh \lambda_1 + \frac{\lambda_1^2}{\lambda_2} \sin \lambda_2\right) \\ c_2 &= \Lambda(\cosh \lambda_1 - \cos \lambda_2) \\ c_3 &= \Lambda\left(\frac{\sinh \lambda_1}{\lambda_1} - \frac{\sin \lambda_2}{\lambda_2}\right) \end{aligned}$$

From Equation (D15), the transfer matrix  $T$  required is thus

$$T = \begin{bmatrix} c_0 - \alpha_2 & \ell[c_1 - (\sigma + \tau) c_3] \\ \frac{\beta^4}{\ell} c_3 & c_0 - \tau c_2 \\ \frac{\beta^3 EJ}{\ell^2} c_2 & \frac{EJ}{\ell} [-c_1 + (\beta^4 + \tau^2) c_3] \\ \frac{\beta^4 EJ}{\ell^3} (c_1 - \alpha_3) & \frac{\beta^4 EJ}{\ell^2} c_2 \end{bmatrix}$$

$$\left. \begin{array}{ll} \frac{\ell^2 c_2}{EJ} & \frac{\ell}{\beta^4 EJ} [-\alpha_1 + (\beta^4 + \sigma^2) c_3] \\ \frac{\ell(c_1 - \tau c_3)}{EJ} & \frac{\ell^2}{EJ} c_2 \\ c_0 - \tau c_2 & \ell[c_1 - (\sigma + \tau) c_3] \\ \frac{\beta^4}{\ell} c_3 & c_0 - \alpha_2 \end{array} \right\}$$

(D16)

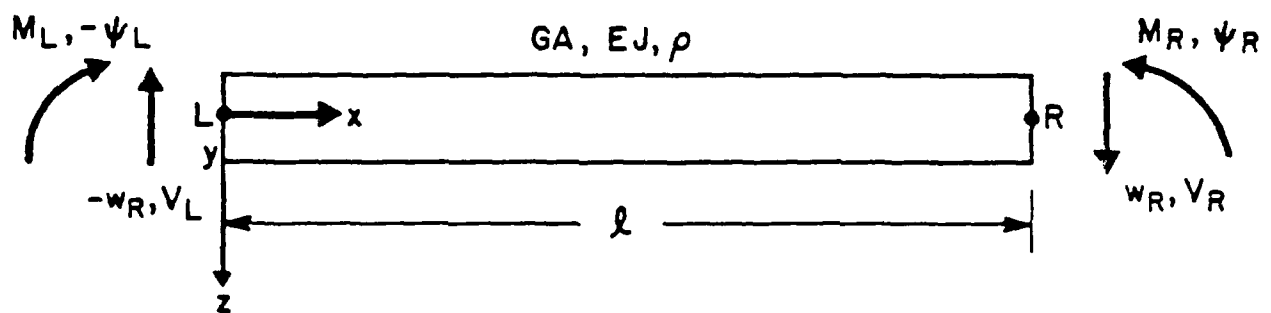


Fig. D1 An elastic bar with distributed mass (Timoshenko beam model).

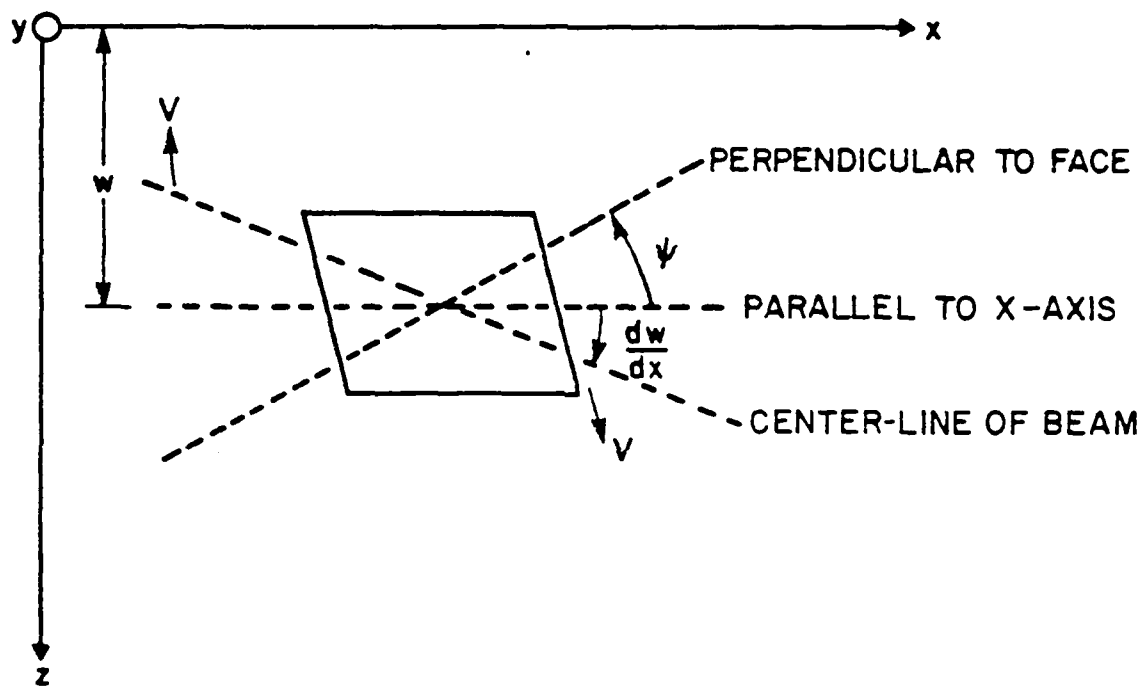
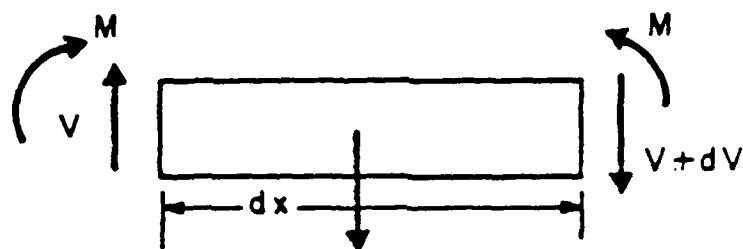


Fig. D2 An element of the Timoshenko beam.



$$\begin{aligned}\text{INERTIA FORCE} &= \mu \omega^2 w \, dx \\ \text{INERTIA COUPLE} &= \mu i^2 \omega^2 \psi \, dx\end{aligned}$$

Fig. D3 Free body diagram of an element of the Timoshenko beam model.

## APPENDIX E

### TRANSFER MATRIX FOR LONGITUDINAL AND FLEXURAL VIBRATION IN AN ELASTIC BAR WITH DISTRIBUTED MASS AND ROTARY INERTIA

The transfer matrix for longitudinal and flexural vibration in an elastic bar with distributed mass and rotary inertia is derived in this appendix. Fig. E1 shows an elastic bar with the sign convention adopted for the forces and displacements in the bar. The bar has modulus of elasticity  $E$ , shear modulus  $G$ , second moment of area inertia about the  $y$  axis  $J$ , radius of gyration about the  $y$  axis  $i$ , cross-sectional area  $A$  and length  $\ell$ . Furthermore, the bar has displacement components which consist of transverse displacement  $w$ , longitudinal displacement  $u$ , rotation of the bar's cross-section  $\psi$  and force components which include the shear force  $V$ , axial force  $N$  and moment  $M$ .

If the transverse deflections are assumed small (relative to the bar cross-section), the longitudinal and flexural vibrations (or waves) are not coupled. The transfer matrix can then be obtained by directly assembling the transfer matrix for longitudinal vibration with the transfer matrix for flexural vibration.

Since for longitudinal vibration in an elastic bar with distributed mass,

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_R = \begin{bmatrix} \cos \theta & \frac{\ell \sin \theta}{EA \theta} \\ -\mu \ell \omega^2 \frac{\sin \theta}{\theta} & \cos \theta \end{bmatrix} \begin{Bmatrix} u \\ N \end{Bmatrix}_L \quad (E1)$$

and for flexural vibration of an elastic bar with distributed mass and rotary inertia,

$$\begin{pmatrix} -w \\ \psi \\ M \\ V \end{pmatrix}_R = \begin{bmatrix} c_0 - \tau c_2 & \ell[c_1 - (\sigma + \tau)c_3] \\ \frac{\beta^4}{\ell} c_3 & c_0 - \tau c_2 \\ \frac{\beta EJ}{\ell^2} c_2 & \frac{EJ}{\ell} [-\tau c_1 + (\beta^4 + \tau^2)c_3] \\ \frac{\beta^4 EJ}{\ell^3} (c_1 - \alpha c_3) & \frac{\beta^4 EJ}{\ell^2} c_2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\ell^2 c_2}{EJ} & \frac{\ell^3}{\beta^4 EJ} [-\alpha c_1 + (\beta^4 + \sigma^2)c_3] \\ \frac{\ell}{EJ} (c_1 - \tau c_3) & \frac{\ell^2}{EJ} c_2 \\ c_0 - \tau c_2 & \ell[c_1 - (\sigma + \tau)c_3] \\ \frac{\beta^4}{\ell} c_3 & c_0 - \tau c_2 \end{bmatrix} \begin{pmatrix} -w \\ \psi \\ M \\ V \end{pmatrix}_L$$

(E2)

Assembling eqns (E1) and (E2),



$$\begin{Bmatrix} u \\ -w \\ \psi \\ M \\ V \\ N \end{Bmatrix}_R = \begin{bmatrix} \cos \theta & 0 & 0 \\ 0 & c_0 - \tau c_2 & \ell [c_1 - (\sigma + \tau) c_3] \\ 0 & \frac{\beta^4}{\ell} c_3 & C_0 - \tau C_2 \\ 0 & \frac{\beta^4 EJ}{\ell^2} c_2 & \frac{EJ}{\ell} [-\tau c_1 + (\beta^4 + \tau^2) c_2] \\ 0 & \frac{\beta^4 EJ}{\ell^3} (c_1 - \alpha_3) & \frac{\beta^4 EJ}{\ell^2} c_3 \\ -\mu \ell \omega^2 \frac{\sin \theta}{\theta} & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & \frac{\ell \sin \theta}{EA \theta} \\ \frac{\ell^3}{\beta^4 EJ} [-\alpha C_1 + (\beta^4 + \sigma^2) C_3] & \frac{\ell^2}{\beta^4 EJ} [-\alpha c_1 + (\beta^4 + \sigma^2) c_3] & 0 \\ \frac{\ell^2}{EJ} c_2 & \frac{\ell^2}{EJ} c_2 & 0 \\ c_0 - \tau c_2 & \ell [c_1 - (\sigma + \tau) c_3] & 0 \\ \frac{\beta^4}{\ell} c_3 & c_0 - \alpha c_2 & 0 \\ 0 & 0 & \cos \theta \end{bmatrix} \begin{Bmatrix} u \\ -w \\ \psi \\ M \\ V \\ N \end{Bmatrix}_L$$

(E3)

$$\text{where } \theta = \ell \omega \sqrt{\frac{\rho}{E}}$$

$$\Lambda = \frac{1}{\lambda_1^2 + \lambda_2^2}$$

$$\lambda_2 = \sqrt{\sqrt{\beta^4 + \frac{1}{4}(\sigma - \tau)^2} \mp \frac{1}{2}(\sigma + \tau)}$$

$$c_0 = \Lambda(\lambda_2 \cosh \lambda_1 + \lambda_1^2 \cos \lambda_2)$$

$$c_1 = \Lambda\left(\frac{\lambda_2}{\lambda_1} \sinh \lambda_1 + \frac{\lambda_1^2}{\lambda_2} \sin \lambda_2\right)$$

$$c_2 = \Lambda(\cosh \lambda_1 - \cos \lambda_2)$$

$$c_3 = \Lambda\left(\frac{\sinh \lambda_1}{\lambda_1} - \frac{\sin \lambda_2}{\lambda_2}\right)$$

$$\sigma = \frac{\mu \omega^2 \ell^2}{GA_s}$$

$$\tau = \frac{\mu^2 \omega^2 \ell^2}{EJ}$$

$$\beta^4 = \frac{\mu \omega^2 \ell^2}{EJ}$$

Thus, the transfer matrix  $T$  specified is

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## WAVE PROPAGATION AND DYNAMICS OF LATTICE STRUCTURES(U)

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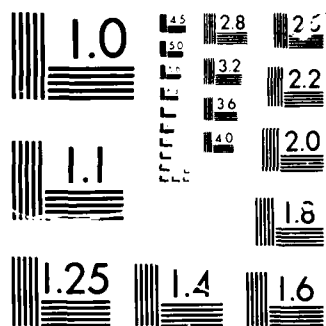
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$$T = \begin{bmatrix} \cos \theta & 0 & 0 \\ 0 & c_0 - \alpha_2 & \ell[c_1 - (\sigma + \tau)c_3] \\ 0 & \frac{\beta^4}{\ell} c_3 & c_0 - \tau c_2 \\ 0 & \frac{\beta^4 EJ}{\ell^2} c_2 & \frac{EJ}{\ell} [-\tau c_1 + (\beta^4 + \tau^2)c_3] \\ 0 & \frac{\beta^4 EJ}{\ell^3} (c_1 - \alpha_3) & \frac{\beta^4 EJ}{\ell} c_2 \\ -\mu \ell \omega^2 \frac{\sin \theta}{\theta} & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & \frac{\ell \sin \theta}{EA \theta} \\ \frac{\ell^2}{EJ} c_2 & \frac{\ell^3}{\beta^4 EJ} [-\alpha_1 + (\beta^4 + \sigma^2)c_3] & 0 \\ \frac{\ell}{EJ} (c_1 - \tau c_3) & \frac{\ell^2}{EJ} c_2 & 0 \\ c_0 - \tau c_2 & \ell[c_1 - (\sigma + \tau)c_3] & 0 \\ \frac{\beta^4}{\ell} c_3 & c_0 - \alpha_2 & 0 \\ 0 & 0 & \cos \theta \end{bmatrix}$$

(E4)

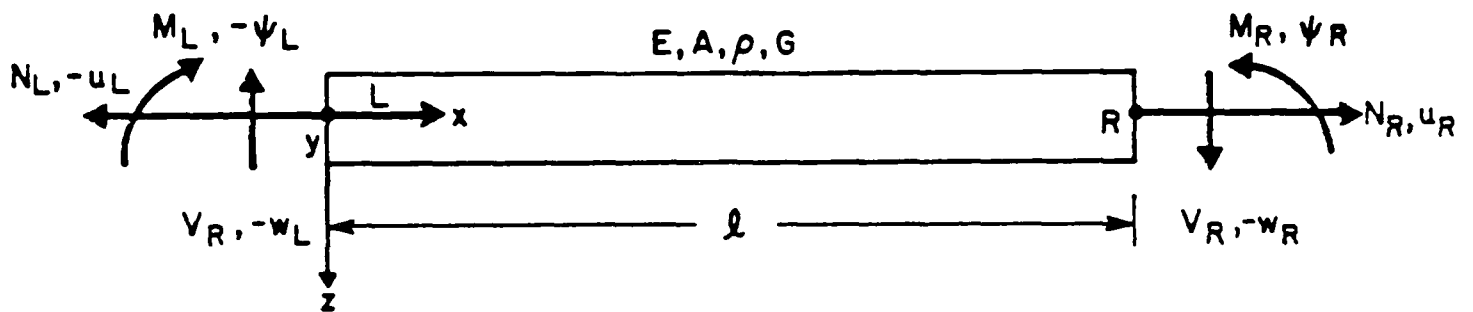


Fig. E1 An elastic bar with distributed mass and rotary inertia, undergoing both longitudinal and flexural deformation.

## APPENDIX F

### FREQUENCY RESPONSE AND IMPULSE RESPONSE FUNCTIONS FOR LONGITUDINAL VIBRATION IN AN ELASTIC ROD

Figure F1 shows the elastic rod to be used in this Appendix. The rod has modulus of elasticity  $E$ , mass density  $\rho$  and cross-sectional area  $A$ .  $N$  denotes the axial force and  $u$  denotes the longitudinal displacement in the rod. For demonstration purposes, the rod is clamped at one end and the other end is left free. The rod is loaded at Section  $E$  with a sinusoidal axial force of magnitude  $N_0$ . Furthermore, the rod is assumed to consist of six identical rod segments, each of length  $\ell$ . In this Appendix, the frequency response functions at sections  $A, B, C, D, E$  and  $F$  will be obtained using transfer matrices. After the frequency response functions are generated, the impulse response functions can be obtained using inverse Fourier transforms. In particular, the impulse response function is obtained for section  $B$ .

#### FREQUENCY RESPONSE FUNCTIONS

For longitudinal vibration in an elastic rod, the state vector  $z$  consists of the longitudinal displacement  $u$  and the internal force  $N$ . Thus,

$$z = \begin{Bmatrix} u \\ N \end{Bmatrix} \quad (F1)$$

At boundary  $A$ , since the displacement is defined (that is,  $u = 0$ ),

$$z_A = \begin{Bmatrix} 0 \\ N \end{Bmatrix}_A \quad (F2)$$

At boundary  $G$ , since the force is defined (that is,  $N = 0$ ),

$$z_G = \begin{Bmatrix} u \\ 0 \end{Bmatrix}_G \quad (F3)$$

The excitation is given by  $Le^{i\omega t}$  where  $L$  is an input vector whose elements correspond to those in the state vector being considered. Thus, in general,

$$L = \begin{Bmatrix} D \\ P \end{Bmatrix} \quad (F4)$$

where  $D$  is the displacement excitation and  $P$  is the force excitation. In the present case, since the axial force at  $E$  is the excitation,

$$Le^{i\omega t} = \begin{Bmatrix} 0 \\ N_o \end{Bmatrix}_E e^{i\omega t} \quad (F5)$$

For longitudinal vibration in an elastic rod, the transfer matrix relates the state vectors at the two ends of a rod segment as, from left to right,

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_R = \begin{bmatrix} \cos \theta & \frac{\ell}{EA} \frac{\sin \theta}{\theta} \\ -\mu \ell \omega^2 \frac{\sin \theta}{\theta} & \cos \theta \end{bmatrix} \begin{Bmatrix} u \\ N \end{Bmatrix}_L \quad (F6)$$

and from right to left,



$$\begin{Bmatrix} u \\ N \end{Bmatrix}_L = \begin{bmatrix} \cos \theta & -\frac{\ell}{EA} \frac{\sin \theta}{\theta} \\ \mu \ell \omega^2 \frac{\sin \theta}{\theta} & \cos \theta \end{bmatrix} \begin{Bmatrix} u \\ N \end{Bmatrix}_R \quad (F7)$$

Let the transfer matrices given in eqn. (F6) and (F7) be  $T(\ell)$  and  $T^{-1}(\ell)$ , respectively, where the  $\ell$  in parentheses signifies that the transfer matrices are for a rod of finite length  $\ell$ . Notice that since there is no impedance mismatch between the individual sections and the rod can be considered as one continuous member, multiplying  $T(\ell)$   $n$  times is equivalent to replacing  $\ell$  in  $T$  by  $n\ell$ , or

$$T^n(\ell) = T(n\ell) \quad (F8)$$

For the state vector at  $G$ ,

$$\begin{Bmatrix} u \\ 0 \end{Bmatrix}_G = T(\ell)T(\ell)T(\ell)T(\ell)T(\ell)T(\ell) \begin{Bmatrix} 0 \\ N \end{Bmatrix}_A + T(\ell)T(\ell) \begin{Bmatrix} 0 \\ N_o \end{Bmatrix}_E \quad (F9)$$

or, using eqn. (F8), eqn. (F9) can be written as

$$\begin{Bmatrix} u \\ 0 \end{Bmatrix}_G = T(6\ell) \begin{Bmatrix} 0 \\ N \end{Bmatrix}_A + T(2\ell) \begin{Bmatrix} 0 \\ N_o \end{Bmatrix}_E \quad (F10)$$

Now let  $A = T(6\ell)$  where  $A$  is a  $2 \times 2$  matrix with elements  $a_{11}$ ,  $a_{12}$ ,  $a_{21}$  and  $a_{22}$  and let  $B = T(2\ell)$  where  $B$  is a  $2 \times 2$  matrix with elements  $b_{11}$ ,  $b_{12}$ ,  $b_{21}$  and  $b_{22}$ . Eqn. (F10) can be written as

$$\begin{Bmatrix} u \\ 0 \end{Bmatrix}_G = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} 0 \\ N \end{Bmatrix}_A + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{Bmatrix} 0 \\ N_o \end{Bmatrix}_E \quad (\text{F11})$$

Considering the force vectors only, from eqn. (F11),

$$0 = a_{22}N_A + b_{22}N_o$$

from which

$$N_A = -\frac{b_{22}}{a_{22}} N_o \quad (\text{F12})$$

$$\text{But since } \theta = \ell\omega\sqrt{\frac{\rho}{E}} \text{ and } n\theta = n\ell\omega\sqrt{\frac{\rho}{E}}$$

examination of eqn. (F6) gives

$$a_{22} = \cos 6\theta \quad (\text{F13})$$

$$\text{and } b_{22} = \cos 2\theta \quad (\text{F14})$$

Substituting eqns. (F13) and (F14) into eqn. (F12),

$$N_A = -\frac{\cos 2\theta}{\cos 6\theta} N_o \quad (\text{F15})$$

Since by definition  $H(\omega) = \frac{\text{response}}{\text{excitation}}$ ,

$${}_N H_A^E(\omega) = - \frac{\cos 2\theta}{\cos 6\theta} \quad (\text{F16})$$

where the superscript  $E$  denotes the excitation location, the subscript  $A$  denotes the response location and the subscript  $N$  denotes a force response. At  $\beta$

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_B = T(\ell) \begin{Bmatrix} 0 \\ N \end{Bmatrix}_A \quad (\text{F17})$$

Let  $c_{11}$ ,  $c_{12}$ ,  $c_{21}$ ,  $c_{22}$  be elements of the  $2 \times 2$  matrix  $T(\ell)$  such that eqn. (F17) becomes

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_B = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{Bmatrix} 0 \\ N \end{Bmatrix}_A$$

or  $\begin{Bmatrix} u \\ N \end{Bmatrix}_B = N_A \begin{bmatrix} c_{12} \\ c_{22} \end{bmatrix} \quad (\text{F18})$

Using eqns. (F15) and (F18),

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_B = - \frac{\cos 2\theta}{\cos 6\theta} N_o \begin{bmatrix} c_{12} \\ c_{22} \end{bmatrix} \quad (\text{F19})$$

Examination of eqn. (F6) gives

$$c_{12} = \frac{\ell}{EA} \frac{\sin \theta}{\theta} \quad (\text{F20})$$

$$c_{22} = \cos \theta \quad (\text{F21})$$

Thus,

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_B = -\frac{\cos 2\theta}{\cos 6\theta} N_o \begin{bmatrix} \frac{\ell}{EA} & \frac{\sin \theta}{\theta} \\ & \cos \theta \end{bmatrix}$$

from which,

$$H_B^F(\omega) = -\frac{\cos 2\theta}{\cos 6\theta} \begin{bmatrix} \frac{\ell}{EA} & \frac{\sin \theta}{\theta} \\ & \cos \theta \end{bmatrix} \quad (F22)$$

Notice that when there is no subscript to the left of  $H(\omega)$ , the frequency response function is a vector whose top element represents the displacement response and whose bottom element represents the force response.

Similarly, for station C ,

$$H_C^F(\omega) = -\frac{\cos 2\theta}{\cos 6\theta} \begin{bmatrix} \frac{2\ell}{EA} & \frac{\sin 2\theta}{2\theta} \\ & \cos 2\theta \end{bmatrix} \quad (F23)$$

for station D ,

$$H_D^F(\omega) = -\frac{\cos 2\theta}{\cos 6\theta} \begin{bmatrix} \frac{3\ell}{EA} & \frac{\sin 3\theta}{3\theta} \\ & \cos 3\theta \end{bmatrix} \quad (F24)$$

and for station  $E_L$  where  $E_L$  is the point just to the left of E ,

$$H_{E_L}^F(\omega) = -\frac{\cos 2\theta}{\cos 6\theta} \begin{bmatrix} \frac{4\ell}{EA} & \frac{\sin 4\theta}{4\theta} \\ & \cos 4\theta \end{bmatrix} \quad (F25)$$

Now the frequency response functions at  $G$ ,  $F$  and  $E_R$  are obtained where  $E_R$  is the point just to the right of  $E$ . To avoid crossing the discontinuity due to the excitation at  $E$ , transfer matrices from right to left are used.

Using Equation (F7), for the state vector at  $A$ ,

$$\begin{Bmatrix} 0 \\ N \end{Bmatrix}_A = T^{-1}(6\ell) \begin{Bmatrix} u \\ 0 \end{Bmatrix}_G - T^{-1}(4\ell) \begin{Bmatrix} 0 \\ N_o \end{Bmatrix}_E \quad (F26)$$

The minus sign is due to the sign convention chosen. This is because transfer matrices from right to left is used. Now let  $D = T^{-1}(6\ell)$  with elements  $d_{11}$ ,  $d_{12}$ ,  $d_{21}$ , and  $d_{22}$  and  $E = T^{-1}(4\ell)$  with elements  $e_{11}$ ,  $e_{12}$ ,  $e_{21}$  and  $e_{22}$ . Equation (F26) can be rewritten as

$$\begin{Bmatrix} 0 \\ N \end{Bmatrix}_A = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{Bmatrix} u \\ 0 \end{Bmatrix}_G - \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \begin{Bmatrix} 0 \\ N_o \end{Bmatrix}_E \quad (F27)$$

Considering only the displacement vector at  $A$ ,

$$0 = d_{11}u_G - e_{12}N_o$$

$$\text{or } u_G = \frac{e_{12}}{d_{11}} N_o \quad (F28)$$

From eqn. (F7)

$$e_{12} = -\frac{4\ell}{EA} \frac{\sin 4\theta}{4\theta} \quad (F29)$$

$$d_{11} = \cos 6\theta \quad (F30)$$

Thus,

$$H_G^u(\omega) = - \frac{\frac{4\ell}{EA} \frac{\sin 4\theta}{4\theta}}{\cos 6\theta} \quad (F31)$$

where the subscript u denotes displacement response. At station F,

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_F = T^{-1}(\ell) \begin{Bmatrix} u \\ 0 \end{Bmatrix}_G \quad (F32)$$

Let  $f_{11}$ ,  $f_{12}$ ,  $f_{21}$  and  $f_{22}$  be the elements of  $T^{-1}(\ell)$ ,

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_F = u_G \begin{bmatrix} f_{11} \\ f_{21} \end{bmatrix} \quad (F33)$$

Using eqns. (F28), (F29), (F30) and (F33),

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_F = - \frac{4\ell}{EA} \frac{\frac{\sin 4\theta}{4\theta}}{\cos 6\theta} \begin{bmatrix} f_{11} \\ f_{21} \end{bmatrix} N_G$$

$$\text{or } H_F^u(\omega) = - \frac{4\ell}{EA} \frac{\frac{\sin 4\theta}{4\theta}}{\cos 6\theta} \begin{bmatrix} f_{11} \\ f_{21} \end{bmatrix} \quad (F34)$$

Since eqn. (F7) gives

$$f_{11} = \cos \theta$$

$$f_{21} = \mu \ell \omega^2 \frac{\sin \theta}{\theta} ,$$

eqn. (F34) can be written as

$$H_F^E(\omega) = -\frac{4\ell}{EA} \frac{\frac{\sin 4\theta}{4\theta}}{\cos 6\theta} \begin{bmatrix} \cos \theta \\ \mu \ell \omega^2 \frac{\sin \theta}{\theta} \end{bmatrix} \quad (F35)$$

Similarly, for station  $E_R$  where  $E_R$  is just to the right of point  $E$ ,

$$H_{E_R}^E(\omega) = -\frac{4\ell}{EA} \frac{\frac{\sin 4\theta}{4\theta}}{\cos 6\theta} \begin{bmatrix} \cos 2\theta \\ 2\mu \ell \omega^2 \frac{\sin 2\theta}{2\theta} \end{bmatrix} \quad (F36)$$

#### IMPULSE RESPONSE FUNCTIONS

In the present case, since the frequency response functions are quite simple (from a mathematical point of view), the impulse response functions can be obtained in closed form by simply taking the inverse Fourier transform of the frequency response functions.

First consider the impulse force response functions at point  $B$ . From Equation (F22), for the force response,

$${}_N H_B^F(\omega) = -\frac{\cos 2\theta}{\cos 6\theta} \cos \theta \quad (F37)$$

where the subscript  $N$  denotes the force response. Using the relationship

$$\cos n\theta = \frac{e^{jn\theta} + e^{-jn\theta}}{2},$$

Equation (F37) can be rewritten as

$${}_N H_B^E(\omega) = - \frac{e^{j2\theta} + e^{-j2\theta}}{2} \cdot \frac{e^{j\theta} + e^{-j\theta}}{2} \cdot \frac{2}{e^{j6\theta} + e^{-j6\theta}} \quad (\text{F38})$$

Rearranging eqn. (F38),

$${}_N H_B^E(\omega) = - \frac{e^{j3\theta} + e^{-j3\theta} + e^{j\theta} + e^{-j\theta}}{2} \cdot \frac{1}{e^{j6\theta} + e^{-j6\theta}} \quad (\text{F39})$$

Multiplying eqn. (F39) by  $\frac{e^{-j6\theta} - e^{j6\theta}}{e^{-j6\theta} - e^{j6\theta}}$  and arranging terms,

$${}_N H_B^E(\omega) = \frac{(e^{j6\theta} - e^{-j6\theta})(e^{j3\theta} + e^{-j3\theta} + e^{j\theta} + e^{-j\theta})}{2} \cdot \frac{1}{e^{-j12\theta} - e^{j12\theta}} \quad (\text{F40})$$

Factoring the term  $e^{-j12\theta}$  from eqn. (F40)

$${}_N H_B^E(\omega) = \frac{e^{j12\theta}}{2} (e^{j6\theta} - e^{-j6\theta})(e^{j3\theta} + e^{-j3\theta} + e^{j\theta} + e^{-j\theta}) \cdot \frac{1}{1 - e^{j24\theta}} \quad (\text{F41})$$

Since  $\frac{1}{1 - e^{j24\theta}} = \sum_{k=0}^{\infty} e^{j24k\theta}$  [13], eqn. (F41) becomes



$${}_N H_B^E(\omega) = \frac{e^{i12\theta}(e^{i6\theta} - e^{-i6\theta})(e^{i3\theta} + e^{-i3\theta} + e^{i\theta} + e^{-i\theta})}{2} \sum_{k=0}^{\infty} e^{i24k\theta} \quad (F42)$$

Rearranging terms in eqn. (F42)

$${}_N H_B^E(\omega) = \frac{1}{2} (e^{i15\theta} + e^{i17\theta} + e^{i19\theta} + e^{i21\theta} - e^{i3\theta} - e^{i5\theta} - e^{i7\theta} - e^{i9\theta}) \sum_{k=0}^{\infty} e^{i24k\theta}$$

or

$${}_N H_B^E(\omega) = \frac{1}{2} \sum_{k=0}^{\infty} (e^{i15\theta} + e^{i17\theta} + e^{i19\theta} + e^{i21\theta} - e^{i3\theta} - e^{i5\theta} - e^{i7\theta} - e^{i9\theta}) e^{i24k\theta} \quad (F43)$$

Since  $h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{i\omega t} d\omega$  [10], and  $\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} e^{i\omega \tau} d\omega = \delta(t - \tau)$  where  $\delta(t - \tau)$  is

the Dirac delta function, also keeping in mind that

$$\theta = \ell \omega \sqrt{\frac{\rho}{E}} \quad ,$$

integration of eqn. (F43) term by term gives

$$\begin{aligned}
 {}_N h_B^E(t) = & \frac{1}{2} \sum_{k=0}^{\infty} \{ -\mathfrak{A}[t - (3+24k)\ell\sqrt{\frac{\rho}{E}}] - \mathfrak{A}[t - (5+24k)\ell\sqrt{\frac{\rho}{E}}] \\
 & - \mathfrak{A}[t - (7+24k)\ell\sqrt{\frac{\rho}{E}}] - \mathfrak{A}[t - (9+24k)\ell\sqrt{\frac{\rho}{E}}] \\
 & + \mathfrak{A}[t - (15+24k)\ell\sqrt{\frac{\rho}{E}}] + \mathfrak{A}[t - (17+24k)\ell\sqrt{\frac{\rho}{E}}] \\
 & + \mathfrak{A}[t - (19+24k)\ell\sqrt{\frac{\rho}{E}}] + \mathfrak{A}[t - (21+24k)\ell\sqrt{\frac{\rho}{E}}] \}
 \end{aligned}$$

(F44)

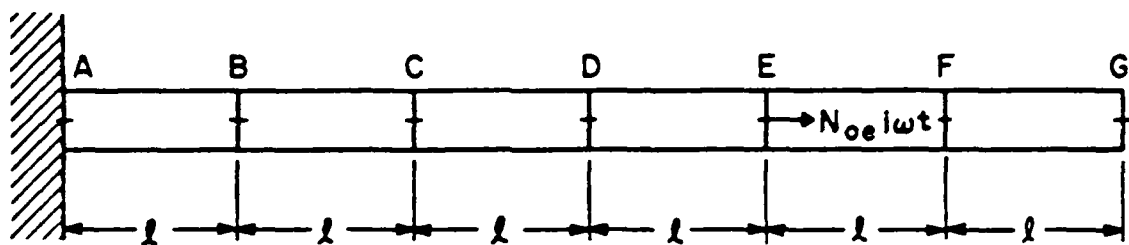


Fig. F1 An elastic rod loaded with a sinusoidal axial force.

## APPENDIX G

### TRANSFER MATRICES FOR WAVE PROPAGATION IN A 3-BAY PLANAR LATTICE STRUCTURE

Fig. G1 shows the 3-bay planar lattice structure to be used in this appendix. The planar structure is assumed to consist of identical elastic bars with distributed mass, each of length  $\ell$ . Each bar has modulus of elasticity  $E$ , mass density  $\rho$ , shear modulus  $G$ ; cross-sectional area  $A$ , second moment of area inertia about the  $y$  axis  $J$  and radius of gyration about the  $y$  axis  $i$ . The cross-sectional dimension of each bar is assumed to be small compared to the length, and the Timoshenko beam model is adopted. Each bar has transverse displacement  $w$ , longitudinal displacement  $u$ , angle of rotation of the cross-section  $\psi$ , axial force  $N$ , shear force  $V$  and moment  $M$ . The sign convention for the displacements and forces in a connecting bar as shown in Fig. G2.

In analyzing the wave propagation and vibration of the planar structure shown in Fig. G1, two transfer matrices are involved. The first transfer matrix  $X_1$  involves the transfer of state vectors in two bars, each of length  $\ell$  in main members I and II. For example, members 12 and 12' together are represented by such a matrix. The second transfer matrix  $X_2$  involves the transfer of state vectors across the junctions. In a periodic unit, the members that join main members I and II constitute such a transfer matrix. For example, member 11' results in a transfer matrix  $X_2$ . Fig. G3 shows a 3-bay planar structure which has been sectioned into its constitutive parts responsible for transfer matrices  $X_1$  and  $X_2$ . The subscripts  $R$  and  $L$  are used to denote points which are just to the right and points which are just to the left, respectively, of junctions which connect main members I and II.

# TRANSFER MATRIX $X_1$

Transfer matrix  $X_1$  can be obtained by simply assembling the transfer matrices for flexural and longitudinal vibrations in straight bars such that both the state vectors in main members I and II are considered. The transfer matrix for longitudinal and flexural vibrations in straight bars is given as

$$\begin{bmatrix} \cos \theta & 0 & 0 \\ 0 & c_0 - \alpha_2 & \ell[c_1 - (\sigma + \tau)c_3] \\ 0 & \frac{\beta^4}{\ell} c_3 & c_0 - \tau c_2 \\ 0 & \frac{\beta^4 EJ}{\ell^2} c_2 & \frac{EJ}{\ell} [-\tau c_1 + (\beta^4 + \tau^2)c_3] \\ 0 & \frac{\beta^4 EJ}{\ell^3} (c_1 - \alpha_3) & \frac{\beta^4 EJ}{\ell^2} c_2 \\ -\mu \ell \omega^2 \frac{\sin \theta}{\theta} & 0 & 0 \\ 0 & 0 & \frac{\ell}{EA} \frac{\sin \theta}{\theta} \\ \frac{\ell^2}{EJ} c_2 & \frac{\ell^3}{\beta^4 EJ} [-\alpha_1 + (\beta^4 + \sigma^2)c_3] & 0 \\ \frac{\ell}{EJ} (c_1 - \tau c_3) & \frac{\ell^2}{EJ} c_2 & 0 \\ c_0 - \tau c_2 & \ell[c_1 - (\sigma + \tau)c_3] & 0 \\ \frac{\beta^4}{\ell} c_3 & c_0 - \alpha_2 & 0 \\ 0 & 0 & \cos \theta \end{bmatrix}$$

corresponding to a state vector

$$z = \begin{pmatrix} d \\ p \end{pmatrix}$$

where

$$d = \begin{Bmatrix} u \\ -w \\ \psi \end{Bmatrix}$$

and

$$p = \begin{Bmatrix} M \\ V \\ N \end{Bmatrix}$$

and where  $\theta = \ell \omega \sqrt{\frac{\rho}{E}}$

$$\Lambda = \frac{1}{\lambda_1^2 + \lambda_2^2}$$

$$\lambda_2 = \sqrt{\sqrt{\beta^4 + \frac{1}{C_1}(\sigma - \tau)^2} \mp \frac{1}{2}(\sigma + \tau)}$$

$$c_0 = \Lambda (\lambda_2 \cosh \lambda_1 + \lambda_1^2 \cos \lambda_2)$$

$$c_1 = \Lambda \left( \frac{\lambda_2}{\lambda_1} \sinh \lambda_1 + \frac{\lambda_1^2}{\lambda_2} \sinh \lambda_2 \right)$$

$$c_2 = \Lambda (\cosh \lambda_1 - \cos \lambda_2)$$

$$c_3 = \Lambda \left( \frac{\sinh \lambda_1}{\lambda_1} - \frac{\sin \lambda_2}{\lambda_2} \right)$$

$$\sigma = \frac{\mu \omega^2 \ell^2}{GA_s}$$

$$\tau = \frac{\mu \omega^2 \ell^2}{GA_s}$$

$$\beta^4 = \frac{\mu \omega^2 \ell^4}{EJ}$$

Partitioning the transfer matrix given into four  $3 \times 3$  submatrices  $C_1, C_2, C_3$ , and  $C_4$  such that

$$C_1 = \begin{bmatrix} \cos \theta & 0 & 0 \\ 0 & c_0 - \alpha_2 & \ell[c_1 - (\sigma + \tau)c_3] \\ 0 & \frac{\beta^4}{\ell} c_3 & c_0 - \tau c_2 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0 & 0 & \frac{\ell}{EA} \frac{\sin \theta}{\theta} \\ \frac{\ell^2}{EJ} c_2 & \frac{\ell^3}{\beta^4 EJ} [-\alpha_1 + (\beta^4 + \sigma^2)c_3] & 0 \\ \frac{\ell}{EJ} (c_1 - \tau c_3) & \frac{\ell^2}{EJ} c_2 & 0 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} 0 & \frac{\beta^4 EJ}{\ell^2} c_2 & \frac{EJ}{\ell} [-\tau c_1 + (\beta^4 + \tau^2)c_3] \\ 0 & \frac{\beta^4 EJ}{\ell^3} (c_1 - \alpha_3) & \frac{\beta^4 EJ}{\ell^2} c_2 \\ -\mu \ell \omega^2 \frac{\sin \theta}{\theta} & 0 & 0 \end{bmatrix}$$

$$C_4 = \begin{bmatrix} c_0 - \tau c_2 & \ell[c_1 - (\sigma + \tau)c_3] & 0 \\ \frac{\beta^4}{\ell} c_3 & c_0 - \alpha_2 & 0 \\ 0 & 0 & \cos \theta \end{bmatrix}$$

The following relations can be written

$$\begin{Bmatrix} d_I \\ p_I \end{Bmatrix}_R = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} \begin{Bmatrix} d_I \\ p_I \end{Bmatrix}_L \quad (G1)$$

$$\begin{Bmatrix} d_{II} \\ p_{II} \end{Bmatrix}_R = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} \begin{Bmatrix} d_{II} \\ p_{II} \end{Bmatrix}_L \quad (G2)$$

where the subscripts I and II stand for member numbers.

Eqs. (G1) and (G2) can be combined to form

$$\begin{Bmatrix} d_I \\ d_{II} \\ p_{II} \\ p_I \end{Bmatrix}_R = \begin{bmatrix} C_1 & 0 & 0 & C_2 \\ 0 & C_1 & C_2 & 0 \\ 0 & C_3 & C_4 & 0 \\ C_3 & 0 & 0 & C_4 \end{bmatrix} \begin{Bmatrix} d_I \\ d_{II} \\ p_{II} \\ p_I \end{Bmatrix}_L \quad (G3)$$

Notice that eqn. (G3) relates the state vectors at the right end to that of the left end of a section of a periodic unit which is represented by transfer matrix  $X_1$ .

From eqn. (G3),

$$X_1 = \begin{bmatrix} C_1 & 0 & 0 & C_2 \\ 0 & C_1 & C_2 & 0 \\ 0 & C_3 & C_4 & 0 \\ C_3 & 0 & 0 & C_4 \end{bmatrix} \quad (G4)$$

TRANSFER MATRIX  $X_2$

From the previous section,

$$\begin{Bmatrix} d \\ p \end{Bmatrix}_R = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} \begin{Bmatrix} d \\ p \end{Bmatrix}_L$$

Consider an I-junction as shown in Fig. G4. The forces and displacements of member 11' (Fig. G4)



are shown in Fig. G5. For member 11',

$$\begin{Bmatrix} \hat{d} \\ \hat{p} \end{Bmatrix}_1 = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} \begin{Bmatrix} d \\ p \end{Bmatrix}_1, \quad (G5)$$

where  $\hat{d}$  and  $\hat{p}$  are displacement and force vectors in the local 11' coordinate.

In crossing over from left to right of junctions 1 and 1', the displacements are unchanged but the bar 11' introduces discontinuities in the forces and moments. Assuming the displacements at the junctions are known, these forces may be computed from the elastic, geometric and mass properties of bar 11'.

Fig. G6 and G7 show the directional relationships between local and global state vectors in junctions 1 and 1', respectively. By inspection of Fig. G6,

$$\begin{aligned} -u_1 &= -\hat{w}_1 ; -w_1 = \hat{u}_1 ; \psi_1 = \hat{\psi}_1 \\ \text{or } \begin{Bmatrix} \hat{u} \\ -\hat{w} \\ \hat{\psi} \end{Bmatrix}_1 &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u \\ -w \\ \psi \end{Bmatrix}_1 \end{aligned} \quad (G6)$$

$$\text{and } \hat{V}_1 = N_1 ; \hat{M}_1 = M_1 ; \hat{N}_1 = -V_1$$

$$\text{or } \begin{Bmatrix} \hat{M} \\ \hat{V} \\ \hat{N} \end{Bmatrix}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} M \\ V \\ N \end{Bmatrix}_1 \quad (G7)$$

Similarly, inspection of Fig. G7 gives

$$-u_{1'} = -\hat{u}_{1'} ; -w_{1'} = \hat{u}_{1'} ; \psi_{1'} = \hat{\psi}_{1'}$$

$$\text{and } \begin{Bmatrix} \hat{u} \\ -\hat{w} \\ \hat{\psi} \end{Bmatrix}_{1'} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u \\ -w \\ \psi \end{Bmatrix}_{1'} \quad (\text{G8})$$

$$\text{and } \hat{V}_{1'} = -N_{1'} ; \hat{M}_{1'} = -M_{1'} ; \hat{N}_{1'} = V_{1'}$$

$$\begin{Bmatrix} M \\ V \\ N \end{Bmatrix}_{1'} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{Bmatrix} \hat{M} \\ \hat{V} \\ \hat{N} \end{Bmatrix}_{1'} \quad (\text{G9})$$

Eqs. (G6) through (G9) can be written as

$$\hat{d} = G_1 d_1 \quad (\text{G10})$$

$$p_1 = G_2 \hat{p}_1 \quad (\text{G11})$$

$$\hat{d}_{1'} = G_3 d_{1'} \quad (\text{G12})$$

$$p_{1'} = G_4 \hat{p}_{1'} \quad (\text{G13})$$

where

$$G_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$G_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G_4 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

From eqn. (G5),

$$\hat{d}_1 = C_1 \hat{d}_1' + C_2 \hat{p}_1' \quad (G14)$$

$$\hat{p}_1 = C_3 \hat{d}_1' + C_4 \hat{p}_1' \quad (G15)$$

Multiplying eqn. (G14) by  $C_2^{-1}$  and rearranging,

$$\hat{p}_1' = C_2^{-1} \hat{d}_1 - C_2^{-1} C_1 \hat{d}_1' \quad (G16)$$

Transforming the local coordinates to the global coordinates in the main members, using eqns. (G10) to (G13),

$$p_1 = G_2 \hat{p}_1$$

$$\begin{aligned} \text{or } p_1 &= G_2 C_4 C_2^{-1} G_1 d_1 \\ &+ G_2 (C_3 - C_4 C_2^{-1} C_1) G_3 d_{1'} \end{aligned} \quad (G17)$$

$$\text{and } p_{1'} = G_4 \hat{p}_{1'}$$

$$\text{or } p_{1'} = G_4 C_2^{-1} G_1 d_1 - G_4 C_2^{-1} C_1 G_3 d_{1'} \quad (G18)$$

Writing eqns. (G17) and (G18) in matrix form,

$$\begin{Bmatrix} p_1 \\ p_{1'} \end{Bmatrix} = \begin{bmatrix} G_2 C_4 C_2^{-1} G_1 & G_2 (C_3 - C_4 C_2^{-1} C_1) G_3 \\ G_4 C_2^{-1} G_1 & -G_4 C_2^{-1} C_1 G_3 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_{1'} \end{Bmatrix} \quad (G19)$$

Now examine Fig. G8, which shows the forces at junction 1. From equilibrium considerations,

$$\begin{Bmatrix} M_1 \\ V_1 \\ N_1 \end{Bmatrix}_R = \begin{Bmatrix} M_1 \\ V_1 \\ N_1 \end{Bmatrix}_L + \begin{Bmatrix} M_1 \\ V_1 \\ N_1 \end{Bmatrix} \quad (G20)$$

where the subscripts  $L$  and  $R$  stand for the left and right end points of 1. Similarly, for junction 1',

$$\begin{Bmatrix} M_{1'} \\ V_{1'} \\ N_{1'} \end{Bmatrix}_R = \begin{Bmatrix} M_{1'} \\ V_{1'} \\ N_{1'} \end{Bmatrix}_L + \begin{Bmatrix} M_{1'} \\ V_{1'} \\ N_{1'} \end{Bmatrix} \quad (G21)$$

From eqns. (G20) and (G21),

$$\begin{Bmatrix} p_1 \\ p_{1'} \end{Bmatrix}_R = \begin{Bmatrix} p_1 \\ p_{1'} \end{Bmatrix}_L + \begin{Bmatrix} p_1 \\ p_{1'} \end{Bmatrix} \quad (G22)$$

Since

$$\begin{Bmatrix} p_1 \\ p_{1'} \end{Bmatrix}_L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{Bmatrix} d_1 \\ d_{1'} \\ p_{1'} \\ p_1 \end{Bmatrix}_L \quad (G23)$$

where  $I$  is the identity matrix, and from eqn. (G19),

$$\begin{Bmatrix} p_1 \\ p_{1'} \end{Bmatrix}_R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ G_4 C_2^{-1} G_1 & -G_4 C_2^{-1} C_1 C_3 & 0 & 0 \\ G_2 C_4 C_2^{-1} G_1 & G_2 (C_{21} - C_4 C_2^{-1} G_1) G_3 & 0 & 0 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_{1'} \\ p_{1'} \\ p_1 \end{Bmatrix}_L \quad (G24)$$

As mentioned previously, displacements are continuous in crossing junctions 1 and 1', thus,

$$\begin{Bmatrix} d_1 \\ d_{1'} \end{Bmatrix}_R = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_{1'} \\ p_{1'} \\ p_1 \end{Bmatrix}_L \quad (G25)$$

Eqs. (G23) through (G25) can be combined to form

$$\begin{Bmatrix} d_1 \\ d_{1'} \\ p_{1'} \\ p_1 \end{Bmatrix}_R = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ G_4 C_2^{-1} G_1 & -G_4 C_2^{-1} C_1 C_3 & I & 0 \\ G_2 C_4 C_2^{-1} G_1 & G_2 (C_{21} - C_4 C_2^{-1} G_1) G_3 & 0 & I \end{bmatrix} \begin{Bmatrix} d_1 \\ d_{1'} \\ p_{1'} \\ p_1 \end{Bmatrix}_L \quad (G26)$$

Keeping in mind that 1 corresponds to a point in member I and 1' corresponds to a point in member II, eqn. (G26) can be rewritten as

$$\begin{Bmatrix} d_I \\ d_{II} \\ p_{II} \\ p_I \end{Bmatrix}_R = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ G_4 C_2^{-1} G_1 & -G_4 C_2^{-1} C_1 C_3 & I & 0 \\ G_2 C_4 C_2^{-1} G_1 & G_2 (C_{21} - C_4 C_2^{-1} G_1) G_3 & 0 & I \end{bmatrix} \begin{Bmatrix} d_I \\ d_{II} \\ p_{II} \\ p_I \end{Bmatrix}_L \quad (G27)$$

Notice that eqn. (G27) relates the state vectors (on main members I and II) on the right end to the state vectors on the left end of a section of a periodic unit which is represented by transfer matrix  $X_2$ .

From eqn. (G27),

$$X_2 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ G_4 C_2^{-1} G_1 & -G_4 C_2^{-1} C_1 C_3 & I & 0 \\ G_2 C_4 C_2^{-1} G_1 & G_2 (C_{21} - C_4 C_2^{-1} G_1) G_3 & 0 & I \end{bmatrix} \quad (G28)$$

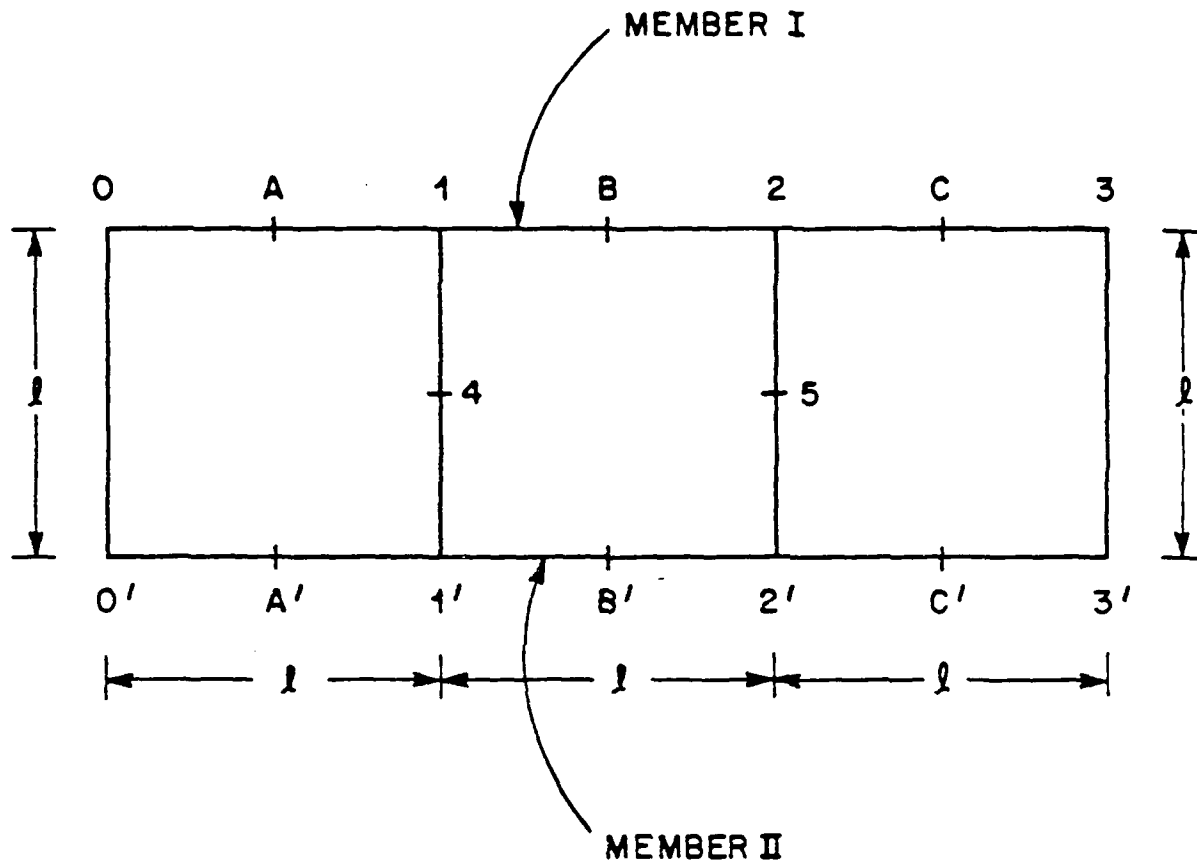


Fig. G1 A 3-bay planar structure.

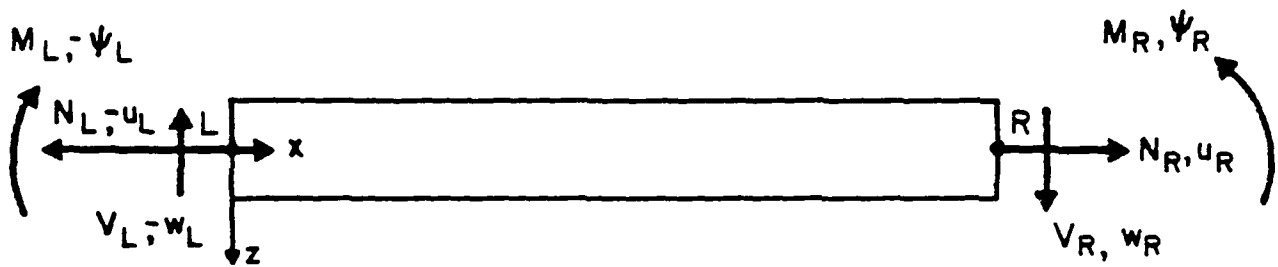


Fig. G2 Sign convention for forces and displacements in a connecting bar.





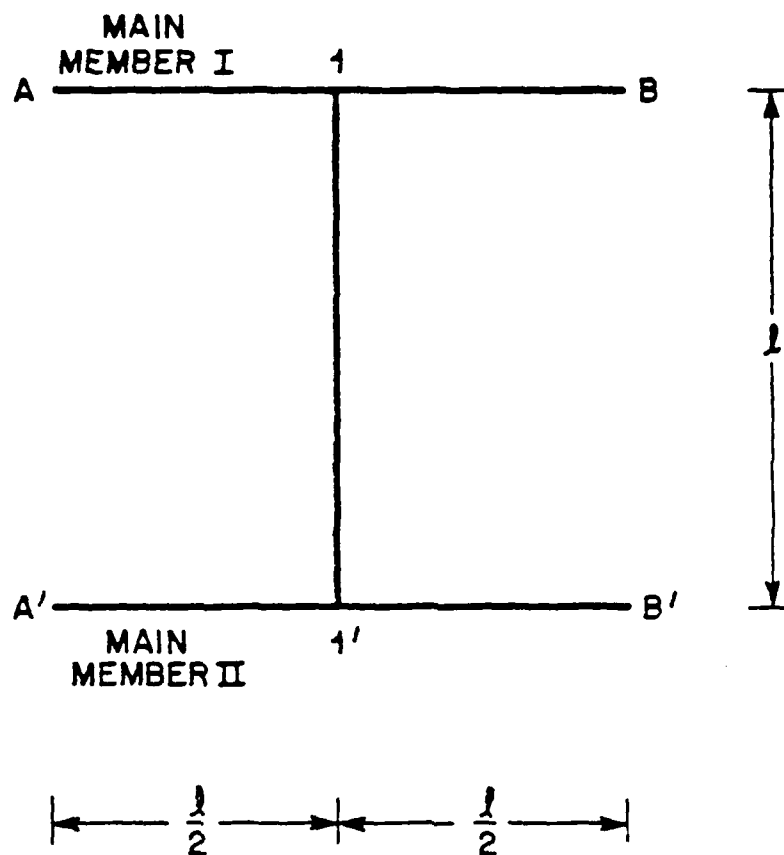


Fig. G4 Diagram of an I-junction.

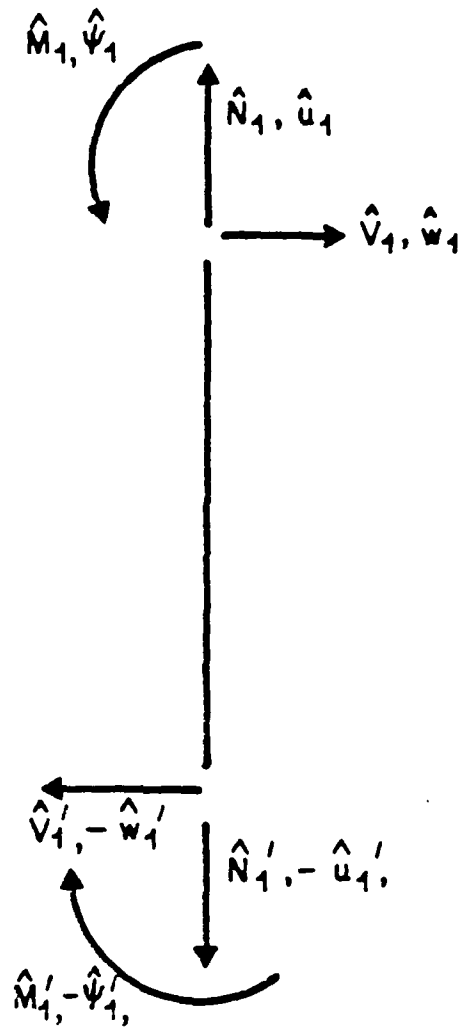


Fig. G5 Free body diagram of member 11'.

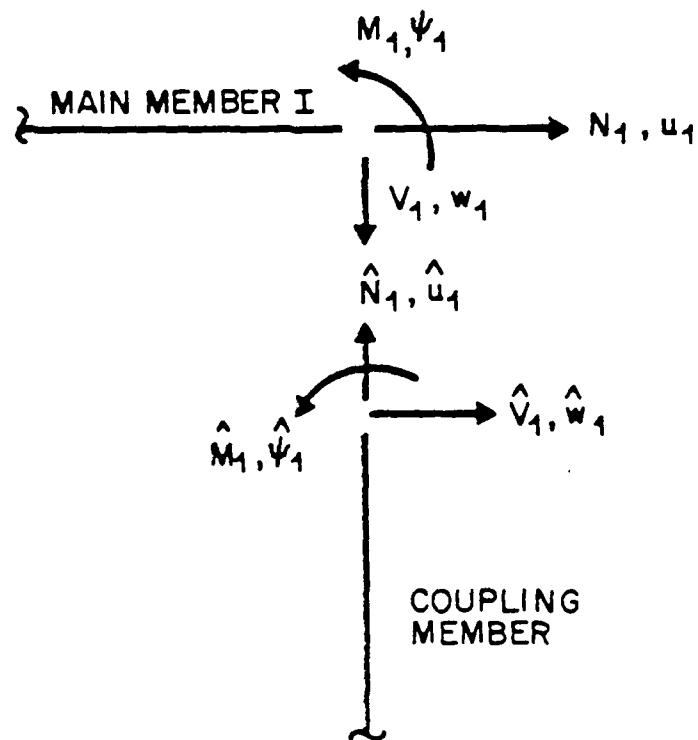


Fig. G6 Directional relations between global and local state vectors at location 1.

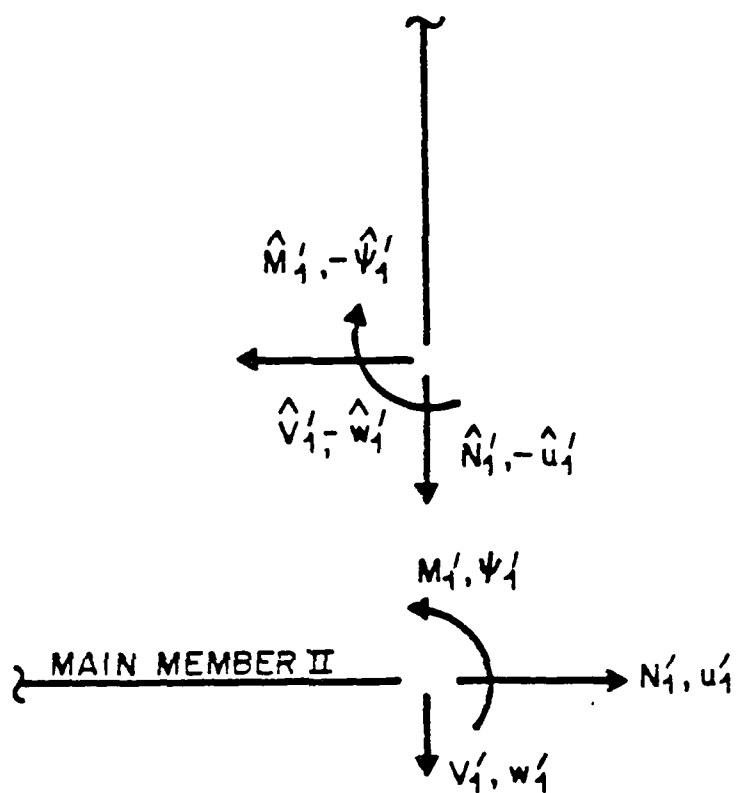


Fig. G7 Directional relations between global and local state vectors at location 1'.

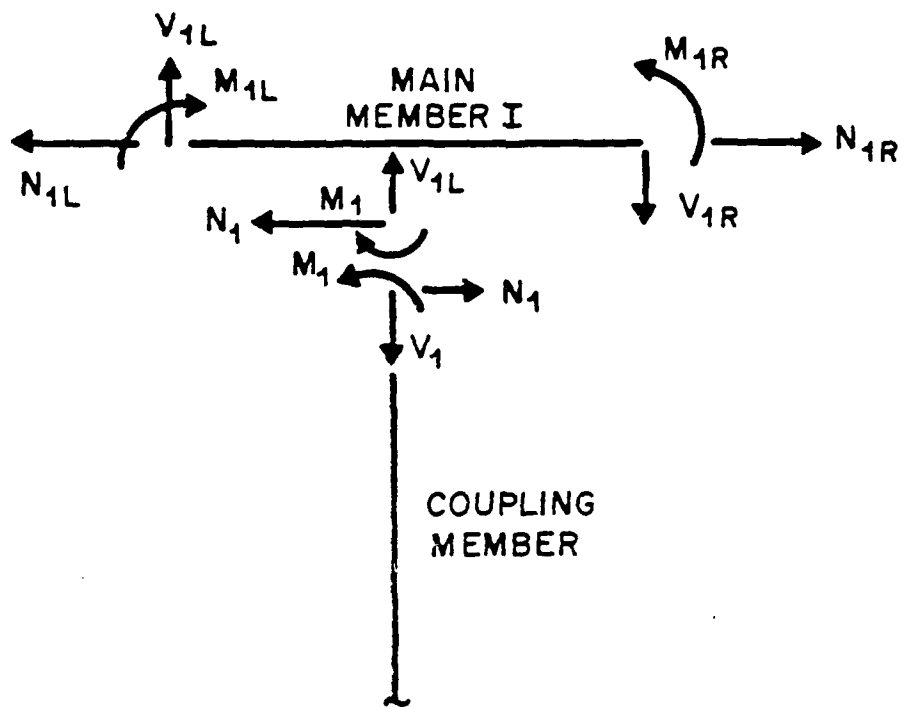


Fig. G8 Forces at location 1.

## APPENDIX H

### FREQUENCY RESPONSE FUNCTIONS FOR A 3-BAY PLANAR LATTICE STRUCTURE

The frequency response functions for specific locations in a planar structure can be obtained through the use of transfer matrices.

Consider the 3-bay planar structure shown in Fig. H1. For demonstration purposes, the structure is loaded at the mid points between members 23 and 2'3' with sinusoidal shear forces. It will be shown that the frequency response functions for locations  $A, A', B, B', 4, 5$  and 6 can be obtained. For simplicity, assume that the cross-sectional dimensions in the connecting bars are small compared to the lengths and that the structure consists of identical rod elements throughout.

The excitation is given by  $Le^{i\omega t}$  where  $L$  is an input vector whose elements correspond to those in the state vector being considered. In other words,

$$L = \begin{Bmatrix} D_I \\ D_{II} \\ P_{II} \\ P_I \end{Bmatrix} \quad (H1)$$

where  $D$  is the displacement excitation vector which includes the longitudinal displacement  $u$ , lateral displacement  $w$  and rotation of the bar cross-sectional area  $\psi$ , and  $P$  is the force excitation vector which includes the moment  $M$ , shear force  $V$  and axial force  $N$ . The subscripts I and II used in eqn. (H1) stand for the entire upper and lower member numbers. In the present case, since the shear forces are the only excitations,

$$Le^{i\omega} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -V \\ 0 \\ 0 \\ 0 \\ V \\ 0 \end{pmatrix} e^{i\omega} \quad (H2)$$

Let

$$L_1 = \begin{pmatrix} 0 \\ V \\ 0 \end{pmatrix}$$

and

$$L_2 = \begin{pmatrix} 0 \\ -V \\ 0 \end{pmatrix}$$

then,

$$L = \begin{pmatrix} 0 \\ 0 \\ L_2 \\ L_1 \end{pmatrix} \quad (H3)$$

where 0 is a null vector of dimension 3.

Since there are no applied forces along the 0-0' and 3-3' boundaries, at 0-0'



$$Z_{0-0'} = \begin{Bmatrix} d_I \\ d_{II} \\ 0 \\ 0 \end{Bmatrix}_{0-0'}$$

and at 3-3'

$$Z_{3-3'} = \begin{Bmatrix} d_I \\ d_{II} \\ 0 \\ 0 \end{Bmatrix}_{3-3'}$$

Now let  $X_1$  be the transfer matrix corresponding to two bar elements of length  $\ell$  in members I and II,  $X^{1/2}$  be the transfer matrix corresponding to two bar elements of length  $\ell/2$  in members I and II, and  $X_2$  be the transfer matrix for the vertical members which connect main members I and II. For the state vector at section 3-3', going from left to right along the structure,

$$\begin{Bmatrix} d_I \\ d_{II} \\ 0 \\ 0 \end{Bmatrix}_{3-3'} = X_2 X_1 X_2 X_1 X_2 X_1 X_2 \begin{Bmatrix} d_I \\ d_{II} \\ 0 \\ 0 \end{Bmatrix}_{0-0'} + X_2 X_1^{1/2} \begin{Bmatrix} 0 \\ 0 \\ L_2 \\ L_1 \end{Bmatrix}_{C-C'} \quad (H4)$$

Let  $K = X_2 X_1 X_2 X_1 X_2 X_1 X_2$  be a 12x12 matrix with elements of 3x3 submatrices where each submatrix is denoted by  $k_{ij}$  ( $i = 1, 2, 3, 4$  and  $j = 1, 2, 3, 4$ ) and let  $B = X_2 X_1^{1/2}$  be a 12x12 matrix with 3x3 submatrices  $b_{ij}$ . Then considering only the load vectors in 3-3', from eqn. (H4),

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}_{3-3'} = \begin{bmatrix} k_{31} & k_{32} \\ k_{41} & k_{42} \end{bmatrix} \begin{Bmatrix} d_I \\ d_{II} \end{Bmatrix}_{0-0'} + \begin{bmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{bmatrix} \begin{Bmatrix} L_2 \\ L_1 \end{Bmatrix}_{C-C'}$$

or

$$\begin{Bmatrix} d_I \\ d_{II} \end{Bmatrix}_{0-0'} = - \begin{bmatrix} k_{31} & k_{32} \\ k_{41} & k_{42} \end{bmatrix}^{-1} \begin{bmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{bmatrix} \begin{Bmatrix} L_2 \\ L_1 \end{Bmatrix}_{C-C'} \quad (H5)$$

At station A-A',

$$\begin{Bmatrix} d_I \\ d_{II} \\ p_I \\ p_{II} \end{Bmatrix}_{A-A'} = X_1^{1/2} X_2 \begin{Bmatrix} d_I \\ d_{II} \\ 0 \\ 0 \end{Bmatrix}_{0-0'}$$

Let  $B' = X_1^{1/2} X_2$  with elements of 3x3 submatrices  $b'_{ij}$ , then

$$\begin{Bmatrix} d_I \\ d_{II} \\ p_{II} \\ p_I \end{Bmatrix}_{A-A'} = \begin{bmatrix} b'_{11} & b'_{12} \\ b'_{21} & b'_{22} \\ b'_{31} & b'_{32} \\ b'_{41} & b'_{42} \end{bmatrix} \begin{Bmatrix} d_I \\ d_{II} \end{Bmatrix}_{0-0'} \quad (H6)$$

Substituting eqn. (H5) into eqn. (H6),

$$\begin{Bmatrix} d_I \\ d_{II} \\ p_{II} \\ p_I \end{Bmatrix}_{A-A'} = - \begin{bmatrix} b'_{11} & b'_{12} \\ b'_{21} & b'_{22} \\ b'_{31} & b'_{32} \\ b'_{41} & b'_{42} \end{bmatrix} \begin{bmatrix} k_{31} & k_{32} \\ k_{41} & k_{42} \end{bmatrix}^{-1} \begin{bmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{bmatrix} \begin{Bmatrix} L_2 \\ L_1 \end{Bmatrix}_{C-C'} \quad (H7)$$

Since  $H(\omega)$  is by definition  $\frac{\text{response}}{\text{excitation}}$ , from eqn. (H7),

$$H_{A-A'}^C(\omega) = - \begin{bmatrix} b'_{11} & b'_{12} \\ b'_{21} & b'_{22} \\ b'_{31} & b'_{32} \\ b'_{41} & b'_{42} \end{bmatrix} \begin{bmatrix} k_{31} & k_{32} \\ k_{41} & k_{42} \end{bmatrix}^{-1} \begin{bmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{bmatrix}$$

where the subscript in  $H(\omega)$  stands for the response location and the superscript stands for the excitation location. Sectioning eqn. (H3),

$$H_A^{C-C'}(\omega) = - \begin{bmatrix} b_{11}' & b_{12}' \\ b_{41}' & b_{42}' \end{bmatrix} \begin{bmatrix} k_{31} & k_{32} \\ k_{41} & k_{42} \end{bmatrix}^{-1} \begin{bmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{bmatrix} \quad (H9)$$

and

$$H_A^{C-C'}(\omega) = - \begin{bmatrix} b_{21}' & b_{22}' \\ b_{31}' & b_{32}' \end{bmatrix} \begin{bmatrix} k_{31} & k_{32} \\ k_{41} & k_{42} \end{bmatrix}^{-1} \begin{bmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{bmatrix} \quad (H10)$$

where the single subscript now denotes a specific point in the structure.

Similarly,

$$H_B^{C-C'}(\omega) = - \begin{bmatrix} b_{11}'' & b_{12}'' \\ b_{41}'' & b_{42}'' \end{bmatrix} \begin{bmatrix} k_{31} & k_{32} \\ k_{41} & k_{42} \end{bmatrix}^{-1} \begin{bmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{bmatrix} \quad (H11)$$

and

$$H_B^{C-C'}(\omega) = - \begin{bmatrix} b_{21}'' & b_{22}'' \\ b_{31}'' & b_{32}'' \end{bmatrix} \begin{bmatrix} k_{31} & k_{32} \\ k_{41} & k_{42} \end{bmatrix}^{-1} \begin{bmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{bmatrix} \quad (H12)$$

where  $b_{ij}''$  are 3x3 submatrices of  $B'' = X_1^{1/2} X_2 X_1 X_2$

To obtain the frequency response functions for stations 4, 5 and 6, transformation matrices are utilized. First consider the forces at junction 1 as shown in Fig. H2. From equilibrium considerations,

$$\begin{Bmatrix} M_1 \\ V_1 \\ N_1 \end{Bmatrix}_1 = \begin{Bmatrix} M_1 \\ V_1 \\ N_1 \end{Bmatrix}_R - \begin{Bmatrix} M_1 \\ V_1 \\ N_1 \end{Bmatrix}_L \quad (H13)$$

Similarly, for junction 1',

$$\begin{Bmatrix} M_1 \\ V_1 \\ N_1 \end{Bmatrix}_1 = \begin{Bmatrix} M_1 \\ V_1 \\ N_1 \end{Bmatrix}_R - \begin{Bmatrix} M_1 \\ V_1 \\ N_1 \end{Bmatrix}_L \quad (\text{H14})$$

where the subscripts  $R$  and  $L$  stand for right and left, respectively, of points 1 and 1'. Eqs. (H13) and (H14) can be combined as

$$\begin{Bmatrix} p_1 \\ p_1 \end{Bmatrix}_1 = \begin{Bmatrix} p_1 \\ p_1 \end{Bmatrix}_R - \begin{Bmatrix} p_1 \\ p_1 \end{Bmatrix}_L \quad (\text{H15})$$

The state vectors to the left and right of junctions 1 and 1' are related by the transfer matrix  $X_2$  such that

$$\begin{Bmatrix} d_1' \\ d_1 \\ p_1' \\ p_1 \end{Bmatrix}_R = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ G_4 C_2^{-1} G_1 & -G_4 C_2^{-1} C_1 G_3 & I & 0 \\ G_2 C_4 C_2^{-1} G_1 & G_2 (C_3 - C_4 C_2^{-1} C_1) G_3 & 0 & I \end{bmatrix} \begin{Bmatrix} d_1' \\ d_1 \\ p_1' \\ p_1 \end{Bmatrix}_L \quad (\text{H16})$$

from which

$$\begin{Bmatrix} p_1' \\ p_1 \end{Bmatrix}_R = \begin{bmatrix} G_4 C_2^{-1} G_1 & -G_4 C_2^{-1} C_1 G_3 & I & 0 \\ G_2 C_4 C_2^{-1} G_1 & G_2 (C_3 - C_4 C_2^{-1} C_1) G_3 & 0 & I \end{bmatrix} \begin{Bmatrix} d_1' \\ d_1 \\ p_1' \\ p_1 \end{Bmatrix}_L \quad (\text{H17})$$

Combining eqns. (H15) and (H17),

$$\begin{Bmatrix} p_1' \\ p_1 \end{Bmatrix}_R = \begin{bmatrix} G_4 C_2^{-1} G_1 & -G_4 C_2^{-1} C_1 G_3 & 0 & 0 \\ G_2 C_4 C_2^{-1} G_1 & G_2 (C_3 - C_4 C_2^{-1} C_1) G_3 & 0 & 0 \end{bmatrix} \begin{Bmatrix} d_1' \\ d_1 \\ p_1' \\ p_1 \end{Bmatrix}_L \quad (\text{H18})$$

Since displacements are continuous in crossing over junctions 1 and 1',

$$\begin{Bmatrix} d_1 \\ d_{1'} \end{Bmatrix} = \begin{Bmatrix} d_1 \\ d_{1'} \end{Bmatrix}_L \quad (\text{H19})$$

Eqs. (H18) and (H19) can be combined as

$$\begin{Bmatrix} d_1 \\ d_{1'} \\ p_{1'} \\ p_1 \end{Bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ G_4 C_2^{-1} G_1 & -G_4 C_2^{-1} C_1 G_3 & 0 & 0 \\ G_2 C_4 C_2^{-1} G_1 & G_2 (C_3 - C_4 C_2^{-1} C_1) G_3 & 0 & 0 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_{1'} \\ p_{1'} \\ p_1 \end{Bmatrix}_L \quad (\text{H20})$$

Keeping in mind that 1 corresponds to member I and 1' corresponds to member II, eqn. (H20) can be rewritten as

$$\begin{Bmatrix} d_I \\ d_{II} \\ p_{II} \\ p_I \end{Bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ G_4 C_2^{-1} G_1 & -G_4 C_2^{-1} C_1 G_3 & 0 & 0 \\ G_2 C_4 C_2^{-1} G_1 & G_2 (C_3 - C_4 C_2^{-1} C_1) G_3 & 0 & 0 \end{bmatrix} \begin{Bmatrix} d_I \\ d_{II} \\ p_{II} \\ p_I \end{Bmatrix}_L \quad (\text{H21})$$

Now let the transfer matrix in eqn. (H21) be  $\bar{X}_2$  for station 1-1',

$$\begin{Bmatrix} d_I \\ d_{II} \\ p_{II} \\ p_I \end{Bmatrix}_{1-1'} = \bar{X}_2 X_1 X_2 \begin{Bmatrix} d_I \\ d_{II} \\ p_{II} \\ p_I \end{Bmatrix}_{0-0'} \quad (\text{H22})$$

Let  $b_{ij}''$  be 3x3 submatrices of  $B'' = \bar{X}_2 X_1 X_2$ , then, since there are no force constraints at station 0-0',

$$\begin{Bmatrix} d_I \\ d_{II} \\ p_{II} \\ p_I \end{Bmatrix}_{1-1'} = \begin{bmatrix} b_{11}'' & b_{12}'' \\ b_{21}'' & b_{22}'' \\ b_{31}'' & b_{32}'' \\ b_{41}'' & b_{42}'' \end{bmatrix} \begin{Bmatrix} d_I \\ d_{II} \end{Bmatrix}_{0-0'} \quad (\text{H23})$$

The directional relationships between the local state vectors in member 11' to that of the global ones in member II at junction point 1' is shown in Fig. H3. Inspection of Fig. H3 gives the following relations,

$$-\hat{u}_{1'} = -u_{1'} \quad \hat{w}_{1'} = -w_{1'} \quad \hat{\psi}_{1'} = \psi_{1'}$$

or

$$\begin{Bmatrix} \hat{u} \\ \hat{w} \\ \hat{\psi} \end{Bmatrix}_{1'} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u \\ -w \\ \psi \end{Bmatrix}_{1'} \quad (\text{H24})$$

$$\text{and } \hat{V}_{1'} = N_{1'} ; \hat{M}_{1'} = -M_{1'} ; \hat{N}_{1'} = -V_{1'}$$

or

$$\begin{Bmatrix} \hat{M} \\ \hat{V} \\ \hat{N} \end{Bmatrix}_{1'} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{Bmatrix} M \\ V \\ N \end{Bmatrix}_{1'} \quad (\text{H25})$$

Combining eqns. (H24) and (H25),

$$\begin{Bmatrix} \hat{d} \\ \hat{p} \end{Bmatrix}_{1'} = \begin{bmatrix} G_1' & 0 \\ 0 & G_2' \end{bmatrix} \begin{Bmatrix} d \\ p \end{Bmatrix}_{1'} \quad (\text{H26})$$

where

$$G_1' = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and where

$$G_2' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

Referring back to Fig. H3, at station 5, which involves the transfer of state vectors in station 1' through a length of  $\ell/2$  from 1' to 5,

$$\begin{Bmatrix} \dot{d} \\ \dot{p} \end{Bmatrix}_5 = T^{1/2} \begin{Bmatrix} \dot{d} \\ \dot{p} \end{Bmatrix}_1,$$

or

$$\begin{Bmatrix} \dot{d} \\ \dot{p} \end{Bmatrix}_5 = T^{1/2} \begin{bmatrix} G_1' & 0 \\ 0 & G_2' \end{bmatrix} \begin{Bmatrix} d \\ p \end{Bmatrix}_1, \quad (\text{H27})$$

where  $T^{1/2}$  is the transfer matrix for a Timoshenko beam of length  $\ell/2$ . Since from eqn. (H23),

$$\begin{Bmatrix} d \\ p \end{Bmatrix}_1 = \begin{bmatrix} b_{21}'' & b_{22}'' \\ b_{31}'' & b_{32}'' \end{bmatrix} \begin{Bmatrix} d_I \\ d_{II} \end{Bmatrix}_{0-0}. \quad (\text{H28})$$

Substitution of eqn. (H28) into eqn. (H27) gives

$$\begin{Bmatrix} \dot{d} \\ \dot{p} \end{Bmatrix}_5 = -T^{1/2} \begin{bmatrix} G_1' & 0 \\ 0 & G_2' \end{bmatrix} \begin{bmatrix} b_{21}'' & b_{22}'' \\ b_{31}'' & b_{32}'' \end{bmatrix} \begin{bmatrix} k_{31} & k_{32} \\ k_{41} & k_{42} \end{bmatrix}^{-1} \begin{bmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{bmatrix} \begin{Bmatrix} L_2 \\ L_1 \end{Bmatrix}_{c-c} \quad (\text{H29})$$

from which

$$H_5^{c-c'}(\omega) = -T^{1/2} \begin{bmatrix} G_1' & 0 \\ 0 & G_2' \end{bmatrix} \begin{bmatrix} b_{21}'' & b_{22}'' \\ b_{31}'' & b_{32}'' \end{bmatrix} \begin{bmatrix} k_{31} & k_{32} \\ k_{41} & k_{42} \end{bmatrix}^{-1} \begin{bmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{bmatrix} \quad (\text{H30})$$

Similarly, for station 4,

$$H_4^C - C'(\omega) = -T^{1/2} \begin{bmatrix} G_1' & 0 \\ 0' & G_2' \end{bmatrix} \begin{bmatrix} b_{21}^N & b_{22}^N \\ b_{31}^N & b_{32}^N \end{bmatrix} \begin{bmatrix} k_{31} & k_{41} \\ k_{32} & k_{42} \end{bmatrix}^{-1} \begin{bmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{bmatrix} \quad (H31)$$

where  $b_{ij}^N$  are 3x3 submatrices of  $B^N = \bar{X}_2$ . For station 6,

$$H_6^C - C'(\omega) = -T^{1/2} \begin{bmatrix} G_1' & 0 \\ 0' & G_2' \end{bmatrix} \begin{bmatrix} b_{21}^V & b_{22}^V \\ b_{31}^V & b_{32}^V \end{bmatrix} \begin{bmatrix} k_{31} & k_{32} \\ k_{41} & k_{42} \end{bmatrix}^{-1} \begin{bmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{bmatrix} \quad (H32)$$

where  $b_{ij}^V$  are 3x3 submatrices of  $B^V = \bar{X}_2 X_1 X_2 X_1$ .

To obtain the state vector at locations to the right of excitation station. C-C', it is more convenient to use transfer matrices from right to left. Since the derivation follows the same procedure outlined previously, it is not discussed further here.



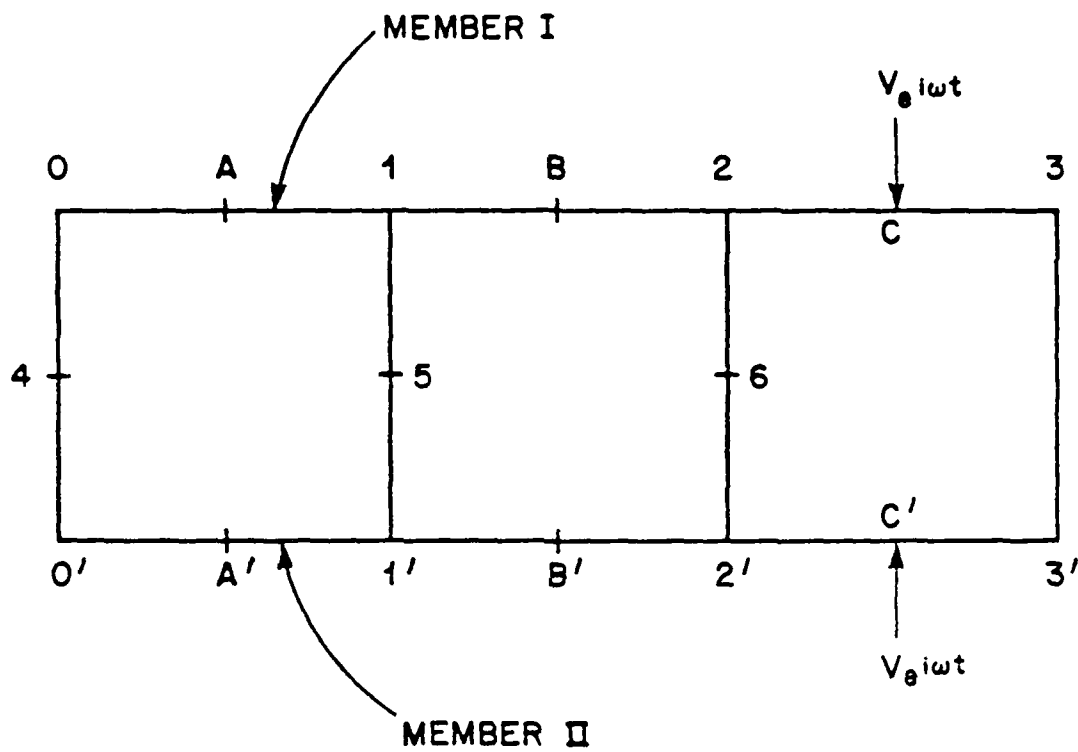


Fig. H1 A 3-pay planar structure loaded with a sinusoidal shear force at C and C'.

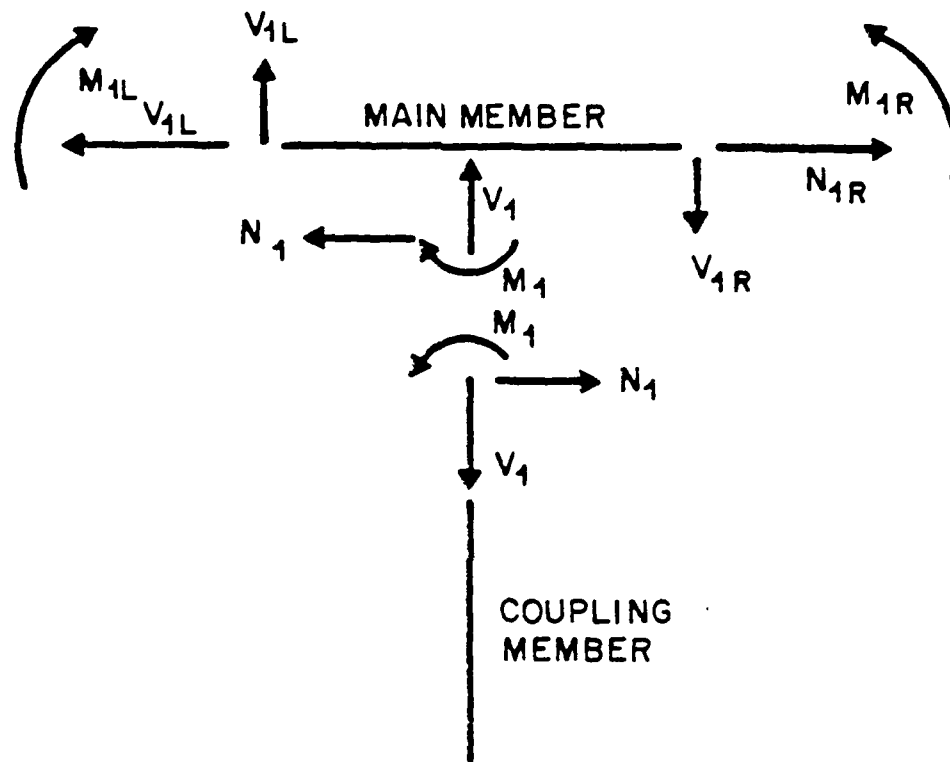


Fig. H2 Forces at location 1.

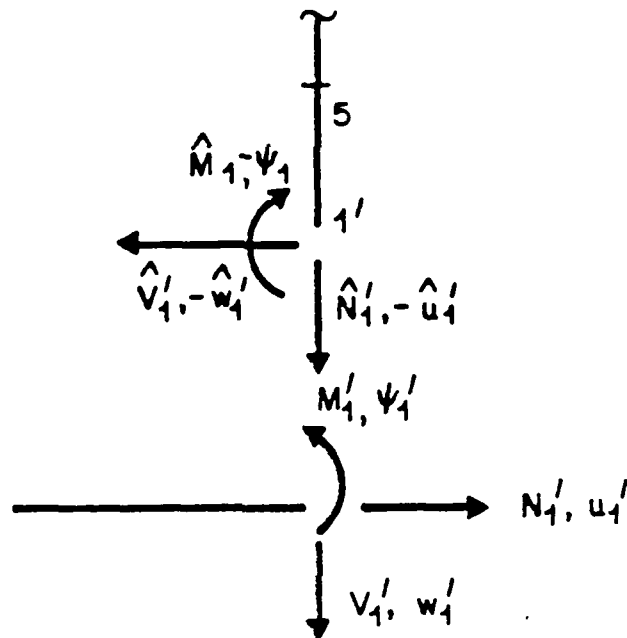


Fig. H3 Directional relations between global and local state vectors at location  $1'$ .

# APPENDIX I

## TRANSFER MATRIX FOR 3-D WAVE PROPAGATION IN A TETRAHEDRAL TRUSS

The transfer matrices for wave propagation in a tetrahedral truss are derived in this appendix. Fig. I1 shows a tetrahedral truss with three repeating periodic units, which is to be used in the derivation of the transfer matrices. The tetrahedral truss is assumed to be made up of identical elastic bars with distributed mass and circular cross sections, each of length  $\ell$ . Each connecting bar has modulus of elasticity  $E$ , mass density  $\rho$ , shear modulus  $G$ , cross-sectional area  $A$ , second moment of area inertia about the  $z$  or  $y$  axis  $J$ , second moment of area inertia about the  $x$  axis  $J_T$ , and radius of gyration about the  $x$  or  $z$  axis  $i$ . The cross-sectional dimension of each connecting bar is assumed to be small compared with its length. Fig. I2 shows the sign conventions for the forces and displacements in a connecting bar in the global  $xyz$  coordinates. Each bar has longitudinal displacement  $u$ , transverse displacements  $v$  and  $w$  in the  $y$  and  $z$  directions, respectively, and angles of rotation  $\phi$ ,  $\psi$  and  $\theta$  about the  $x$ ,  $y$  and  $z$  axis, respectively. In addition, each bar also has moments  $M_y$  and  $M_z$  about the  $y$  and  $z$  axes, respectively, torsion  $T$ , axial force  $N$  and shear forces  $V_y$  and  $V_z$  along the  $y$  and  $z$  axes, respectively.

In the analysis of wave propagation in a tetrahedral truss (Fig. I1), three transfer matrices, namely,  $V_1$ ,  $V_2$  and  $V_3$  are developed. Fig. I3 shows an arbitrary segment of the tetrahedral truss and Fig. I4 shows the segment which is sectioned along planes parallel to the  $yz$  planes into the substructures responsible for the transfer matrices  $V_1$ ,  $V_2$  and  $V_3$ . Referring to Fig. I4, the sections are made by cutting the periodic unit (Fig. I3) along planes  $GA_RID_R$ ,  $E_RK_RF_RL_R$  (or  $E_LK_LF_LL_L$ ) and  $HB_LJC_L$  which are all parallel to the  $yz$  planes. The subscripts  $L$  and  $R$  used for points  $E$ ,  $A$ ,  $D$ ,  $F$ ,  $K$ ,  $L$ ,  $B$  and  $C$  denote points just left and right of these points, respectively. Thus, the first transfer matrix  $V_1$  represents the transfer of state vectors in members which join

members I and III. For example, members AD and BC each result in a transfer matrix  $V_1$ . The second transfer matrix  $V_2$  involves the transfer of state vectors in four bars, each of length  $\ell/2$  in members I through IV together with members which connect member I to member II, member III to member II, member I to member IV and member III to member IV. Referring to Fig. I4, this includes members  $A_R K_L, I F_L, D_R L_L, G_1 E_L, A_R F_L, D_R F_L, A_R E_L, D_R E_L$ . Lastly, transfer matrix  $V_3$  is responsible for the transfer of state vectors in four bars, each of length  $\ell/2$  in members I and III together with members which connect member II to member I, member II to member III, member IV to member I and member IV to member III. For example, referring again to Fig. I4, this includes members  $K_R B_L, F_R J, L_R C_L, E_R H, F_R B_L, F_R C_L, E_R B_L$  and  $E_R C_L$ .

#### TRANSFER MATRIX $V_1$

For the flexural and longitudinal 3-D vibration in an elastic bar, corresponding to a state

vector  $z$  where  $z = \begin{Bmatrix} d \\ p \end{Bmatrix}$

and

$$d = \begin{Bmatrix} U \\ V \\ -W \\ \phi \\ \psi \\ \theta \end{Bmatrix}, \quad p = \begin{Bmatrix} M_x \\ M_y \\ T \\ V_x \\ V_y \\ N \end{Bmatrix}$$

the transfer matrix  $T$  is given as [5],

$$T = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} \quad (11)$$

where  $C_1, C_2, C_3$  and  $C_4$  are  $6 \times 6$  submatrices such that

$$C_1 = \begin{bmatrix} \cos \theta & 0 & 0 & 0 & 0 & 0 \\ 0 & c_0 - \alpha_2 & 0 & 0 & 0 & \ell[c_1 - (\sigma + \tau)c_3] \\ 0 & 0 & c_0 - \alpha_2 & 0 & \ell[c_1 - (\sigma + \tau)c_3] & 0 \\ 0 & 0 & 0 & \cos \alpha & 0 & 0 \\ 0 & 0 & \frac{\beta^4}{\ell}c_3 & 0 & c_0 - \tau c_1 & 0 \\ 0 & \frac{\beta^4}{\ell}c_3 & 0 & 0 & 0 & c_0 - \tau c_2 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0 & 0 & 0 \\ ac_2 & 0 & 0 \\ 0 & ac_2 & 0 \\ 0 & 0 & \frac{1}{GJ_T} \frac{\sin \alpha}{\alpha} \\ 0 & \frac{a}{\ell}(c_1 - \tau c_3) & 0 \\ \frac{a}{\ell}(c_1 - \tau c_3) & 0 & 0 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} 0 & 0 & \frac{\ell}{EA} \frac{\sin \theta}{\theta} \\ 0 & a \frac{\ell}{\beta^4} [-\alpha_1 + c_3^4 + \sigma^2]c_3 & 0 \\ a \frac{\ell}{\beta^4} [-\alpha_1 - (\beta^4 + \sigma^2)c_3] & 0 & 0 \\ 0 & 0 & 0 \\ ac_2 & 0 & 0 \\ 0 & ac_2 & 0 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} 0 & \frac{\beta^4}{a} c_2 & 0 \\ 0 & 0 & \frac{\beta^4}{a} c_2 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\beta^4}{a \ell} (c_1 - \alpha_3) \\ 0 & \frac{\beta^4}{a} \ell (c_1 - \alpha_3) & 0 \\ \mu \ell \omega^2 \frac{\sin \theta}{\theta} & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & \frac{\ell}{a} [-\alpha_1 + (\beta^4 + \tau^2) c_3] \\ 0 & \frac{1}{a} [-\tau c_1 + (\beta^4 + \tau^2) c_3] & 0 \\ \frac{-\alpha \tau G}{e} \sin \alpha & 0 & 0 \\ 0 & \frac{\beta^4}{a} c_2 & 0 \\ 0 & 0 & \frac{\beta^4}{a} c_2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_4 = \begin{bmatrix} c_0 - \tau c_3 & 0 & 0 & 0 & \ell [c_1 - (\sigma + \tau) c_3] & 0 \\ 0 & c_0 - \tau c_3 & 0 & \ell [c_1 - (\sigma + \tau) c_3] & 0 & 0 \\ 0 & 0 & \cos \theta & 0 & 0 & 0 \\ 0 & \frac{\beta^4}{\ell} c_3 & 0 & c_0 - \alpha_2 & 0 & 0 \\ \frac{\beta^4}{\ell} c_3 & 0 & 0 & 0 & c_0 - \alpha_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos \theta \end{bmatrix}$$

$$\text{where } \theta = \ell \omega \sqrt{\frac{\rho}{E}}$$

$$\alpha = \ell \omega \sqrt{\frac{\rho}{G}}$$

$$a = \frac{\ell}{EJ}$$

$$\beta^4 = \frac{\mu \omega^2 \ell^4}{GA_s}$$

$$\sigma = \frac{\mu \omega^2 \ell^2}{GA_s}$$

$$\tau = \frac{\mu^2 \omega^2 \ell^2}{EJ}$$

$$\lambda_{1/2} = \sqrt{\sqrt{\beta^4 + \frac{1}{C_1}(\sigma - \tau)^2} \mp \frac{1}{2}(\sigma + \tau)}$$

$$\Lambda = \frac{1}{\lambda_1^2 + \lambda_2^2}$$

$$C_0 = \Lambda (\lambda_2 \cosh \lambda_1 + \lambda_1^2 \cos \lambda_2)$$

$$C_1 = \Lambda \left( \frac{\lambda_2}{\lambda_1} \sinh \lambda_1 + \frac{\lambda_1^2}{\lambda_2} \sin \lambda_2 \right)$$

$$C_2 = \Lambda (\cosh \lambda_1 - \cos \lambda_2)$$

$$C_3 = \Lambda \left( \frac{\sinh \lambda_1}{\lambda_1} - \frac{\sin \lambda_2}{\lambda_2} \right)$$

Consider member *AD* (Fig. 13) which is representative of transfer matrix  $V_1$ . Fig. 15 shows the directional relationships between the global state vectors and the local state vectors.



Inspection of Fig. 15 gives the following relations,

$$\hat{u} = -v ; \hat{v} = u ; -\hat{w} = -w$$

$$\hat{\phi} = -\psi ; \hat{\psi} = \phi ; \hat{\theta} = \theta$$

$$\hat{M}_x = M_x ; \hat{M}_y = T ; \hat{T} = -M_y$$

$$\hat{V}_x = V_x ; \hat{V}_y = N ; \hat{N} = -V_y$$

In matrix form, for junction D,

$$\begin{pmatrix} \hat{u} \\ \hat{v} \\ -\hat{w} \\ \hat{\phi} \\ \hat{\psi} \\ \hat{\theta} \end{pmatrix}_D = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} u \\ v \\ -w \\ \phi \\ \psi \\ \theta \end{pmatrix}_D \quad (12)$$

and

$$\begin{pmatrix} \hat{M}_x \\ \hat{M}_y \\ \hat{T} \\ \hat{V}_x \\ \hat{V}_y \\ \hat{N} \end{pmatrix}_D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{pmatrix} M_x \\ M_y \\ T \\ V_x \\ V_y \\ N \end{pmatrix}_D \quad (13)$$

Similarly, for junction A,

$$\begin{pmatrix} \hat{u} \\ \hat{v} \\ -\hat{w} \\ \hat{\phi} \\ \hat{\psi} \\ \hat{\theta} \end{pmatrix}_A = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} u \\ v \\ -w \\ \phi \\ \psi \\ \theta \end{pmatrix}_A \quad (14)$$

and

$$\begin{pmatrix} \hat{M}_x \\ \hat{M}_y \\ \hat{T} \\ \hat{V}_x \\ \hat{V}_y \\ \hat{N} \end{pmatrix}_A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} M_x \\ M_y \\ T \\ V_x \\ V_y \\ N \end{pmatrix}_A \quad (15)$$

Eqs. (12) to (15) can be rewritten as

$$\hat{d}_D = G_1 \hat{d}_D \quad (16)$$

$$\hat{p}_D = G_2 \hat{p}_D \quad (17)$$

$$\hat{d}_A = G_3 \hat{d}_A \quad (18)$$

$$\hat{p}_A = G_4 \hat{p}_A \quad (19)$$

where  $G_1 = G_3$  such that

$$G_1 = G_3 = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (I10)$$

and

$$G_2 = -G_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \quad (I11)$$

For member  $AD$ ,

$$\begin{Bmatrix} \hat{d} \\ \hat{p} \end{Bmatrix}_D = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} \begin{Bmatrix} \hat{d} \\ \hat{p} \end{Bmatrix}_A \quad (I12)$$

where  $C_1, C_2, C_3$  and  $C_4$  are defined previously.

From eqn. (I12),

$$\hat{d}_D = C_1 \hat{d}_A + C_2 \hat{p}_A \quad (I13)$$

$$\hat{p}_D = C_3 \hat{d}_A + C_4 \hat{p}_A \quad (I14)$$

Multiplying eqn. (I13) by  $C_2^{-1}$ ,

$$\hat{p}_A = C_2^{-1} \hat{d}_D - C_2^{-1} C_1 \hat{d}_A \quad (I15)$$

Substituting eqn. (I15) into eqn. (I14),

$$\hat{p}_D = C_4 C_2^{-1} \hat{d}_D + (C_3 - C_4 C_2^{-1} C_1) \hat{d}_A \quad (I16)$$

Using eqns. (I15), (I16) and eqns. (I8) and (I7),

$$p_A = G_4 C_2^{-1} G_1 d_D - G_4 C_2^{-1} C_1 G_3 d_A \quad (I17)$$

Using eqns. (I16) and (I6) and eqns. (I8) and (I9),

$$p_D = G_2 C_4 C_2^{-1} G_1 d_D + G_2 (C_3 - C_4 C_2^{-1} C_1) G_3 d_A \quad (I18)$$

Eqns. (I17) and (I18) give the force response at A and at D due to the displacements at A and D.

Now consider Fig. I6, which shows the forces at junctions A and D. From equilibrium considerations,

$$\begin{Bmatrix} p_A \\ p_D \end{Bmatrix}_R = \begin{Bmatrix} p_A \\ p_D \end{Bmatrix} + \begin{Bmatrix} p_A \\ p_D \end{Bmatrix}_L \quad (I19)$$

where the subscripts *L* and *R* denote points just left and just right of A or D.

Keeping in mind that A is in member I and D is in member III, using eqns. (I17), (I18) and (I19),

$$\begin{Bmatrix} p_{IV} \\ p_{III} \\ p_{II} \\ p_I \end{Bmatrix}_R = \begin{bmatrix} 0 & 0 & 0 & I & 0 & 0 & 0 \\ G_2(C_3 - C_4 C_2^{-1} C_1) G_3 & 0 & G_2 C_4 C_2^{-1} G_1 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ -G_4 C_2^{-1} C_1 G_3 & 0 & G_4 C_2^{-1} G_1 & 0 & 0 & 0 & I \end{bmatrix} \begin{Bmatrix} p_{IV} \\ p_{III} \\ p_{II} \\ p_I \end{Bmatrix}_L$$

(I20)

where the subscripts I, II, III and IV denote member numbers.

In crossing the joints A and D, since displacements are continuous,

$$\begin{Bmatrix} d_I \\ d_{II} \\ d_{III} \\ d_{IV} \end{Bmatrix}_R = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{Bmatrix} d_I \\ d_{II} \\ d_{III} \\ d_{IV} \end{Bmatrix}_L$$

(I21)

Combining eqns. (I20) and (I21),

$$\begin{Bmatrix} d_I \\ d_{II} \\ d_{III} \\ d_{IV} \\ p_{IV} \\ p_{III} \\ p_{II} \\ p_I \\ p_{III} \end{Bmatrix}_R = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ G_2(C_3 - C_4 C_2^{-1} C_1) G_3 & 0 & G_2 C_4 C_2^{-1} G_1 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ -G_4 C_2^{-1} C_1 G_3 & 0 & G_4 C_2^{-1} G_1 & 0 & 0 & 0 & 0 & I & 0 \\ G_2(C_3 - C_4 C_2^{-1} C_1) G_3 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \end{bmatrix} \begin{Bmatrix} d_I \\ d_{II} \\ d_{III} \\ d_{IV} \\ p_{IV} \\ p_{III} \\ p_{II} \\ p_I \\ p_{III} \end{Bmatrix}_L$$

(I22)

Thus, from eqn. (I22),

$$V_1 = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ G_2(G_3 - C_4 C_2^{-1} C_1) G_3 & 0 & G_2 C_4 C_2^{-1} G_1 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ -G_4 C_2^{-1} C_1 G_3 & 0 & G_4 C_2^{-1} G_1 & 0 & 0 & 0 & 0 & I \end{bmatrix}$$

(123)

#### TRANSFER MATRIX $V_2$

Fig. I7 shows a substructure which is representative of transfer matrix  $V_2$ . Consider first members  $A_R K_L$ ,  $I F_L$ ,  $D_R L_L$  and  $G E_L$ . Notice that these members are parts of main members I, II, III and IV; respectively, each of length  $l/2$ .

From eqn. (I1), the transfer matrix  $T$  for longitudinal and flexural vibration in a bar of length  $l$  is given by

$$T = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$$

where the submatrices  $C_1$ ,  $C_2$ ,  $C_3$  and  $C_4$  were defined previously. Thus, for a bar of length  $l/2$ , the transfer matrix  $T'$  is given by

$$T' = \begin{bmatrix} C'_1 & C'_2 \\ C'_3 & C'_4 \end{bmatrix} \quad (124)$$

where the submatrices  $C'_1$ ,  $C'_2$ ,  $C'_3$  and  $C'_4$  are obtained by substituting  $l/2$  for  $l$  in submatrices  $C_1$ ,  $C_2$ ,  $C_3$  and  $C_4$ , respectively.

Using eqn. (I24), the following relationships are obtained,

$$\begin{Bmatrix} d \\ p \end{Bmatrix}_{K_L} = \begin{bmatrix} C_1' & C_2' \\ C_3' & C_4' \end{bmatrix} \begin{Bmatrix} d \\ p \end{Bmatrix}_{A_R} \quad (I25)$$

$$\begin{Bmatrix} d \\ p \end{Bmatrix}_{F_L} = \begin{bmatrix} C_1' & C_2' \\ C_3' & C_4' \end{bmatrix} \begin{Bmatrix} d \\ p \end{Bmatrix}_{I_L} \quad (I26)$$

$$\begin{Bmatrix} d \\ p \end{Bmatrix}_{L_L} = \begin{bmatrix} C_1' & C_2' \\ C_3' & C_4' \end{bmatrix} \begin{Bmatrix} d \\ p \end{Bmatrix}_{D_R} \quad (I27)$$

$$\begin{Bmatrix} d \\ p \end{Bmatrix}_{E_L} = \begin{bmatrix} C_1' & C_2' \\ C_3' & C_4' \end{bmatrix} \begin{Bmatrix} d \\ p \end{Bmatrix}_{G_L} \quad (I28)$$

Keeping in mind that  $A_R$ ,  $I$ ,  $D_R$  and  $G$  are all left-end points of the substructure (shown in Fig. I7) in numbers I, II, III and IV, respectively and  $K_L$ ,  $F_L$ ,  $L_L$  and  $E_L$  are all right end points of the substructure in members I, II, III and IV, respectively, and using eqns. (I25) to (I28).

$$\begin{Bmatrix} d_I \\ d_{II} \\ d_{III} \\ d_{IV} \\ p_{IV} \\ p_{III} \\ p_{II} \\ p_I \end{Bmatrix}_{M_R} = \begin{bmatrix} C_1' & 0 & 0 & 0 & 0 & 0 & 0 & C_2' \\ 0 & C_1' & 0 & 0 & 0 & 0 & C_2' & 0 \\ 0 & 0 & C_1' & 0 & 0 & C_2' & 0 & 0 \\ 0 & 0 & 0 & C_1' & C_2' & 0 & 0 & 0 \\ 0 & 0 & 0 & C_3' & C_4' & 0 & 0 & 0 \\ 0 & 0 & C_3' & 0 & 0 & C_4' & 0 & 0 \\ 0 & C_3' & 0 & 0 & 0 & 0 & C_4' & 0 \\ C_3' & 0 & 0 & 0 & 0 & 0 & 0 & C_4' \end{bmatrix} \begin{Bmatrix} d_I \\ d_{II} \\ d_{III} \\ d_{IV} \\ p_{IV} \\ p_{III} \\ p_{II} \\ p_I \end{Bmatrix}_{M_L} \quad (I29)$$

where the subscripts  $M_R$  and  $M_L$  denote the right-end and left-end points of the main members only.

Now consider members  $A_R F_L$ ,  $D_R F_L$ ,  $A_R E$  and  $D_R E_L$ , which are the remaining

members to be analyzed. Notice that these members, each of length  $l$ , join the main members together and the local state vector coordinate of each does not coincide with the global state vector coordinates in the four main members. Thus, with respect to each member, transformation matrices have to be utilized in order to consider the transfer of state vectors from one point in a main member to another point in another main member (for example, from  $A_R$  in main member I to  $F_L$  in main member II through member  $A_R F_L$ ).

Consider first member  $A_R F_L$ . Fig. I8 shows the orientation of member  $A_R F_L$  in the global  $xyz$  coordinate. The directional relationships between the state vectors in the local coordinates  $\hat{x}\hat{y}\hat{z}$  and in the global coordinates  $xyz$  at  $f_L$  are shown in Fig. I9. Notice that the local  $\hat{x}\hat{y}\hat{z}$  axes are chosen such that the  $\hat{y}$  axis is in the  $yz$  plane. To find the transformation matrices needed to convert the state vectors from the local coordinates to those of the global ones, first assume that the  $xyz$  coordinates is rotated about the  $x$ -axis to a new  $x'y'z'$  coordinates such that now the new  $y'$ -axis coincides with the local  $\hat{y}$  axis. Notice that member  $A_R F_L$  is now in the  $x'y'$  plane. Fig. I10 shows the directional relations between the local and the rotated global state vectors at  $F_L$ . Examination of Fig. I10 gives the following relations:



$$\hat{u} = (\cos 60^\circ)(u) + (-\cos 30^\circ)(-w)$$

$$\text{or } \hat{u} = \left(\frac{1}{2}\right)(u) + \left(-\frac{\sqrt{3}}{2}\right)(-w) , \quad (I30)$$

$$-\hat{w} = (\cos 30^\circ)(u) + (-\cos 60^\circ)(-w)$$

$$\text{or } -\hat{w} = \left(-\frac{\sqrt{3}}{2}\right)(u) + \left(\frac{1}{2}\right)(-w) , \quad (I31)$$

$$\hat{\phi} = (\cos 60^\circ)(\phi) + (-\cos 30^\circ)(-\theta)$$

$$\text{or } \hat{\phi} = \left(\frac{1}{2}\right)(\phi) + \left(-\frac{\sqrt{3}}{2}\right)(-\theta) , \quad (I32)$$

$$\hat{\theta} = (-\cos 30^\circ)(\theta) + (\cos 60^\circ)(-\theta)$$

$$\text{or } \hat{\theta} = \left(-\frac{\sqrt{3}}{2}\right)(\theta) + \left(\frac{1}{2}\right)(-\theta) , \quad (I33)$$

$$\text{and } M'_s = (\cos 60^\circ)(\dot{M}_s) + (-\cos 30^\circ)(\dot{T})$$

$$\text{or } M'_s = \left(\frac{1}{2}\right)(\dot{M}_s) + \left(-\frac{\sqrt{3}}{2}\right)(\dot{T}) , \quad (I34)$$

$$T' = (-\cos 30^\circ)(\dot{M}_s) + (\cos 60^\circ)(\dot{T})$$

$$\text{or } T' = \left(-\frac{\sqrt{3}}{2}\right)(\dot{M}_s) + \left(\frac{1}{2}\right)(\dot{T}) , \quad (I35)$$

$$V'_s = (\cos 60^\circ)(\dot{V}_s) + (-\cos 30^\circ)(\dot{T})$$

$$\text{or } V'_s = \left(\frac{1}{2}\right)(\dot{V}_s) + \left(-\frac{\sqrt{3}}{2}\right)(\dot{T}) , \quad (I36)$$

$$N' = (-\cos 30^\circ)(\dot{V}_s) + (\cos 60^\circ)(\dot{T})$$

$$\text{or } N' = \left(-\frac{\sqrt{3}}{2}\right)(\dot{V}_s) + \left(\frac{1}{2}\right)(\dot{T}) , \quad (I37)$$

Using eqns. (I30) to (I33),

$$\begin{Bmatrix} \dot{u} \\ \dot{v} \\ -\dot{w} \\ \dot{\phi} \\ \dot{\psi} \\ \dot{\theta} \end{Bmatrix}_{FL} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} u' \\ v' \\ -w' \\ \phi' \\ \psi' \\ \theta' \end{Bmatrix}_{FL} \quad (I38)$$

and using eqns. (I34) to (I37),

$$\begin{pmatrix} M_s' \\ M_y' \\ T' \\ V_s' \\ V_y' \\ N' \end{pmatrix}_{r_L} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{pmatrix} \hat{M}_s \\ \hat{M}_y \\ \hat{T} \\ \hat{V}_s \\ \hat{V}_y \\ \hat{N} \end{pmatrix}_{r_L} \quad (I39)$$

To obtain the relationships between the unrotated global state vectors and the local state vectors, the  $x'y'z'$  coordinates are rotated back to the original  $xyz$  position. This can be done using another set of transformation matrices. Fig. I11 shows the relationship between the global state vectors and the rotated state vectors. As before, examination of Fig. I10 gives

$$\begin{pmatrix} u' \\ v' \\ -w' \\ \phi' \\ \psi' \\ \theta' \end{pmatrix}_{r_L} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ 0 & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{pmatrix} u \\ v \\ -w \\ \phi \\ \psi \\ \theta \end{pmatrix}_{r_L} \quad (I40)$$

and

$$\begin{pmatrix} M_y \\ M_z \\ T \\ V_x \\ V_y \\ V_z \\ N \end{pmatrix}_{F_L} = \begin{bmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} M'_x \\ M'_y \\ T' \\ V'_x \\ V'_y \\ V'_z \\ N' \end{pmatrix}_{F_L} \quad (I41)$$

Combining eqns. (I38) and (I40),

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ -\dot{w} \\ \dot{\phi} \\ \dot{\psi} \\ \dot{\theta} \end{pmatrix}_{F_L} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ 0 & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{pmatrix} u \\ v \\ -w \\ \phi \\ \psi \\ \theta \end{pmatrix}_{F_L}$$

from which

$$\left\{ \begin{array}{c} \hat{u} \\ \hat{v} \\ -\hat{w} \\ \hat{\phi} \\ \hat{\psi} \\ \hat{\theta} \end{array} \right\}_{F_L} = \left[ \begin{array}{cccccc} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & \sqrt{\frac{1}{3}} & \frac{1}{2}\sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2}\sqrt{\frac{2}{3}} & \frac{1}{2}\sqrt{\frac{1}{3}} \end{array} \right] \left\{ \begin{array}{c} u \\ v \\ -w \\ \phi \\ \psi \\ \theta \end{array} \right\}_{F_L}$$

(I42)

Combining eqns. (I39) and (I41),

$$\left\{ \begin{array}{c} M_y \\ M_z \\ T \\ V_y \\ V_z \\ N \end{array} \right\}_{F_L} = \left[ \begin{array}{cccccc} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \left[ \begin{array}{cccccc} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{array} \right] \left\{ \begin{array}{c} \dot{M}_y \\ \dot{M}_z \\ \dot{T} \\ \dot{V}_y \\ \dot{V}_z \\ \dot{N} \end{array} \right\}_{F_L}$$

from which

$$\begin{Bmatrix} M_y \\ M_z \\ \hat{T} \\ \hat{V}_x \\ \hat{V}_y \\ \hat{N} \end{Bmatrix}_{F_L} = \begin{bmatrix} \frac{1}{2}\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ -\frac{1}{2}\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & -\frac{1}{2}\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} \hat{M}_y \\ \hat{M}_z \\ \hat{T} \\ \hat{V}_x \\ \hat{V}_y \\ \hat{N} \end{Bmatrix}_{F_L} \quad (I43)$$

Similarly, consideration of junction  $A_R$  gives,

$$\begin{Bmatrix} u \\ v \\ -w \\ \phi \\ \psi \\ \theta \end{Bmatrix}_{A_R} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2}\sqrt{\frac{1}{3}} & \frac{1}{2}\sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{11}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2}\sqrt{\frac{2}{3}} & \frac{1}{2}\sqrt{\frac{1}{3}} \end{bmatrix} \begin{Bmatrix} u \\ v \\ -w \\ \phi \\ \psi \\ \theta \end{Bmatrix}_{A_R} \quad (I44)$$

$$\begin{Bmatrix} M_x \\ M_y \\ T \\ V_x \\ V_y \\ N \end{Bmatrix}_{F_L} = \begin{bmatrix} \frac{1}{2}\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ -\frac{1}{2}\sqrt{\frac{1}{3}} & -\sqrt{\frac{2}{3}} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{\frac{1}{3}} & -\sqrt{\frac{2}{3}} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{bmatrix} \begin{Bmatrix} M_x \\ M_y \\ T \\ V_x \\ V_y \\ N \end{Bmatrix}_{F_L} \quad (I45)$$

Eqs. (I42) to (I45) can be rewritten as

$$\dot{d}_{FL} = G'_1 d_{FL} \quad (I46)$$

$$p_{FL} = G'_2 \dot{p}_{FL} \quad (I47)$$

$$\dot{d}_{AR} = G'_3 d_{AR} \quad (I48)$$

$$p_{AR} = G'_4 \dot{p}_{AR} \quad (I49)$$

where

$$G'_1 = G'_3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2}\sqrt{\frac{1}{3}} & \frac{1}{2}\sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2}\sqrt{\frac{2}{3}} & \frac{1}{2}\sqrt{\frac{1}{3}} \end{bmatrix}$$

and where

$$G_2' = -G_4' = \begin{bmatrix} \frac{1}{2}\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ -\frac{1}{2}\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & -\frac{1}{2}\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

From eqn. (11), in the local  $\hat{x}\hat{y}\hat{z}$  coordinate,

$$\begin{Bmatrix} \hat{d} \\ \hat{p} \end{Bmatrix}_{FL} = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} \begin{Bmatrix} \hat{d} \\ \hat{p} \end{Bmatrix}_{AR} \quad (150)$$

Thus, from eqn. (150),

$$\hat{d}_{FL} = C_1 \hat{d}_{AR} + C_2 \hat{p}_{AR} \quad (151)$$

$$\text{and } \hat{p}_{FL} = C_3 \hat{d}_{AR} + C_4 \hat{p}_{AR} \quad (152)$$

Multiplying eqn. (151) by  $C_2^{-1}$ ,

$$\hat{p}_{AR} = C_2^{-1} \hat{d}_{FL} - C_2^{-1} C_1 \hat{d}_{AR} \quad (153)$$

Substituting eqn. (153) into eqn. (152),



$$\dot{p}_{FL} = C_4 C_2^{-1} \dot{d}_{FL} + (C_3 - C_4 C_2^{-1} C_1) \dot{d}_{AR} \quad (I54)$$

Using eqns. (I53), (I46), (I48) and (I49)

$$p_{AR} = G_4' C_2^{-1} G_1' d_{FL} - G_4' C_2' C_1 G_3' d_{AR} \quad (I55)$$

Using eqns. (I54), (I46), (I48) and (I45),

$$p_{FL} = G_2' C_4 C_2^{-1} G_1' d_{FL} + G_2' (C_3 - C_4 C_2^{-1} C_1) G_3' d_{AR} \quad (I56)$$

If the same procedure is carried out for members  $D_R F_L$ ,  $A_R E_L$  and  $D_R E_L$ , and if for the analysis of each member, taking the left end point and the origin of the local coordinate and the direction along the length of the member as the local  $\hat{x}$  direction, with the  $\hat{y}$  axis in the global  $yz$  plane, the following relations are obtained.

For member  $D_R F_L$ ,

$$p_{DR} = G_4' C_2^{-1} G_1' d_{FL} - G_4' C_2^{-1} C_1 G_3' d_{DR} \quad (I57)$$

$$p_{FL} = G_2' C_4 C_2^{-1} G_1' d_{FL} + G_2' (C_3 - C_4 C_2^{-1} C_1) G_3' d_{DR} \quad (I58)$$

where

$$G_1' = G_3' = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\sqrt{\frac{1}{3}} & -\frac{1}{2}\sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{3}} \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2}\sqrt{\frac{1}{3}} & -\frac{1}{2}\sqrt{\frac{2}{3}} \end{bmatrix}$$

and where

$$G_{12}' = G_4' = \begin{bmatrix} -\frac{1}{2}\sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{3}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ -\frac{1}{2}\sqrt{\frac{1}{3}} & -\sqrt{\frac{2}{3}} & -\frac{1}{2} & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{3}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & -\frac{1}{2}\sqrt{\frac{1}{3}} & -\sqrt{\frac{2}{3}} & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

For member  $A_R E_L$ ,

$$p_{AR} = G_4'' C_2^{-1} G_1'' d_{EL} - G_4'' C_2^{-1} C_1 G_3'' d_{AR} \quad (159)$$

$$p_{EL} = G_2'' C_4 C_2^{-1} G_1'' d_{EL} + G_2'' (C_3 - C_4 C_2^{-1} C_1) G_3'' d_{AR} \quad (160)$$

where

$$G_1'' = G_3'' = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2}\sqrt{\frac{1}{3}} & -\frac{1}{2}\sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{3}} \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2}\sqrt{\frac{1}{3}} & -\frac{1}{2}\sqrt{\frac{2}{3}} \end{bmatrix}$$

and where

$$G_2'' = -G_4'' = \begin{bmatrix} \frac{1}{2}\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{2}\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & -\frac{1}{2}\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

For member  $D_R E_L$ ,

$$p_{D_R} = G_4'' C_2^{-1} G_1'' d_{E_L} - G_4'' C_2^{-1} C_1 G_3'' d_{D_R} \quad (161)$$

$$p_{E_L} = G_2'' C_4 C_2^{-1} G_1'' d_{E_L} + G_2'' (C_3 - C_4 C_2^{-1} C_1) G_3'' d_{D_R} \quad (162)$$

where

$$G_1'' = G_3'' = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\sqrt{\frac{1}{3}} & \frac{1}{2}\sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2}\sqrt{\frac{2}{3}} & \frac{1}{2}\sqrt{\frac{1}{3}} \end{bmatrix}$$

and where  $G_2'' = -G_4''$

$$= \begin{bmatrix} \frac{1}{2}\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ -\frac{1}{2}\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & -\frac{1}{2}\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

Eqs. (155) to (162) give the relations between the induced forces at the end points of member  $A_R F_L, D_R F_L, A_R E_L$  and  $D_R E_L$  due to displacements at  $A_R$  and  $F_L, D_R$  and  $F_L, A_R$  and  $E_L, D_R$  and  $E_L$ , respectively. Keeping in mind that  $A_R$  refers to a point in main member I;  $D_R$  refers to a point in main member III,  $E_L$  refers to a point in main member IV, and  $F_L$  refers to a point in main member II, and introducing a set of matrices  $D_1$  through  $D_{16}$  such that

$$D_1 = G_4' C_2^{-1} G_1'$$

$$D_2 = -G_4' C_2^{-1} C_1 G_3'$$

$$D_3 = G_2' C_4 C_2^{-1} G_1'$$

$$D_4 = G_2' (C_3 - C_4 C_2^{-1} C_1) G_3'$$

$$D_5 = G_4'' C_2^{-1} G_1''$$

$$D_6 = -G_4'' C_2^{-1} C_1 G_3''$$

$$D_7 = G_2'' C_4 C_2^{-1} G_1''$$

$$D_8 = G_2'' (C_3 - C_4 C_2^{-1} C_1) G_3''$$

$$D_9 = G_4''' C_2^{-1} G_1'''$$

$$D_{10} = -G_4''' C_2^{-1} C_1 G_3'''$$

$$D_{11} = G_2''' C_4 C_2^{-1} G_1'''$$

$$D_{12} = G_2''' (C_3 - C_4 C_2^{-1} C_1) G_3'''$$

$$D_{13} = G_4''' C_2^{-1} G_1'''$$

$$D_{14} = -G_4''' C_2^{-1} C_1 G_3'''$$

$$D_{15} = G_2''' C_4 C_2^{-1} G_1'''$$

and  $D_{16} = G_2''' (C_3 - C_4 C_2^{-1} C_1) G_3'''$ , eqns. (I55) through (I64) can be expressed in matrix form as

$$\begin{pmatrix} d_I \\ d_{II} \\ d_{III} \\ d_{IV} \\ p_{IV} \\ p_{III} \\ p_{II} \\ p_I \end{pmatrix}_{C_R} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ D_{12} & 0 & D_{16} & D_{11}+D_{15} & 0 & 0 & 0 & 0 \\ 0 & D_5 & D_6+D_{14} & D_{13} & 0 & 0 & 0 & 0 \\ D_4 & D_3+D_7 & D_8 & 0 & 0 & 0 & 0 & 0 \\ D_2+D_{10} & D_1 & 0 & D_9 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} d_I \\ d_{II} \\ d_{III} \\ d_{IV} \\ p_{IV} \\ p_{III} \\ p_{II} \\ p_I \end{pmatrix}_{C_L} \quad (163)$$

where the subscripts  $C_L$  and  $C_R$  denote the left and right ends of the connecting members which join the main members in a section of a periodic unit which results in transfer matrix  $V_2$ . Now, adding the contributions from members  $A_R K_L$ ,  $IF_L$ ,  $D_R L_L$  and  $G_1 E_L$  using eqn. (129), to that in eqn. (163), transfer matrix  $V_L$  can be obtained. Notice that the addition of the elements in the matrices in eqns. (163) and (129) is in effect adding the contributions from the constitutive members which make up transfer matrix  $V_1$ . Thus,

$$\begin{pmatrix} d_I \\ d_{II} \\ d_{III} \\ d_{IV} \\ p_{IV} \\ p_{III} \\ p_{II} \\ p \end{pmatrix}_R = \begin{pmatrix} C'_1 & 0 & 0 & 0 & 0 & 0 & 0 & C'_2 \\ 0 & C'_1 & 0 & 0 & 0 & 0 & 0 & C'_2 \\ 0 & 0 & C'_1 & 0 & 0 & 0 & 0 & C'_2 \\ 0 & 0 & 0 & C'_1 & 0 & 0 & 0 & C'_2 \\ D_{12} & 0 & D_{16} & D_{11}+D_{15}+C'_3 & C'_4 & 0 & 0 & 0 \\ 0 & D_5 & D_6+D_{14}+C'_3 & D_{13} & 0 & C'_4 & 0 & 0 \\ D_4 & D_3+D_7+C'_3 & D_8 & 0 & 0 & 0 & C'_4 & 0 \\ D_2+D_{10}+C'_3 & D_1 & 0 & D_9 & 0 & 0 & 0 & C'_4 \end{pmatrix} \begin{pmatrix} d_I \\ d_{II} \\ d_{III} \\ d_{IV} \\ p_{IV} \\ p_{III} \\ p_{II} \\ p_I \end{pmatrix}_L$$

from which

$$V_2 = \begin{bmatrix} C_1' & 0 & 0 & 0 & 0 & 0 & 0 & C_2' \\ 0 & C_1' & 0 & 0 & 0 & 0 & C_2' & 0 \\ 0 & 0 & C_1' & 0 & 0 & C_2' & 0 & 0 \\ 0 & 0 & 0 & C_1' & C_2' & 0 & 0 & 0 \\ D_{12} & 0 & D_{16} & D_{11}+D_{15}+C_3' C_4' & 0 & 0 & 0 & 0 \\ 0 & D_5 & D_6+D_{14}+C_3' & D_{13} & 0 & C_4' & 0 & 0 \\ D_4 & D_3+D_7+C_3' & D_8 & 0 & 0 & 0 & C_4' & 0 \\ D_2+D_{10}+C_3' & D_1 & 0 & D_9 & 0 & 0 & 0 & C_4' \end{bmatrix} \quad (I64)$$

where the subscripts  $R$  and  $L$  denote the right and left ends of the members in a section of a periodic structure which results in transfer matrix  $V_2$ .

#### TRANSFER MATRIX $V_3$

Fig. I12 shows a substructure which is representative of the transfer matrix which is representative of transfer matrix  $V_3$ . The derivation of the transfer matrix  $V_3$  is similar to that of the transfer matrix  $V_2$ . Referring to Fig. I12, members  $K_R B_L$ ,  $F_R J$ ,  $L_R C_L$  and  $E_R H$ , which are parts of main members I, II, III and IV, respectively, are considered first. Members  $F_R B_L$ ,  $F_R C_L$ ,  $E_R B_L$  and  $E_R C_L$  are then considered. The transfer matrix  $V_3$  is then obtained by adding the contributions of all eight members, as was done for transfer matrix  $V_2$ . Since the procedure is similar to that of the previous section, it will not be repeated here. Transfer matrix  $V_3$  is obtained as

$$V_3 = \begin{bmatrix} C_1' & 0 & 0 & 0 & 0 & 0 & 0 & C_2' \\ 0 & C_1' & 0 & 0 & 0 & 0 & C_2' & 0 \\ 0 & 0 & C_1' & 0 & 0 & C_2' & 0 & 0 \\ 0 & 0 & 0 & C_1' & C_2' & 0 & 0 & 0 \\ E_9 & 0 & E_{13} & E_{10}+E_{14}+C_3' C_4' & 0 & 0 & 0 & 0 \\ 0 & E_8 & E_7+E_{15}+C_3' & E_{16} & 0 & C_4' & 0 & 0 \\ E_1 & E_2+E_6+C_3' & E_5 & 0 & 0 & 0 & C_4' & 0 \\ E_3+E_{11}-C_3' & E_4 & 0 & E_{12} & 0 & 0 & 0 & C_4' \end{bmatrix} \quad (I65)$$

where

$$E_1 = H_4' C_2^{-1} H_1'$$

$$E_2 = -H_4' C_2^{-1} C_1 H_3'$$

$$E_3 = H_2' C_4 C_2^{-1} H_1'$$

$$E_4 = H_2' (C_3 - C_4 C_2^{-1} C_1) H_3'$$

$$E_5 = H_4' C_2^{-1} H_1''$$

$$E_6 = -H_4' C_2^{-1} C_1 H_3''$$

$$E_7 = H_2' C_4 C_2^{-1} H_1''$$

$$E_8 = H_2' (C_3 - C_4 C_2^{-1} C_1) H_3''$$

$$E_9 = H_4'' C_2^{-1} H_1'''$$

$$E_{10} = -H_4'' C_2^{-1} C_1 H_3'''$$

$$E_{11} = H_2'' C_4 C_2^{-1} H_1'''$$

$$E_{12} = H_2'' (C_3 - C_4 C_2^{-1} C_1) H_3'''$$

$$E_{13} = H_4''' C_2^{-1} H_1''''$$

$$E_{14} = -H_4''' C_2^{-1} C_1 H_3''''$$

$$E_{15} = H_2''' C_4 C_2^{-1} H_1''''$$

$$E_{16} = H_2''' (C_3 - C_4 C_2^{-1} C_1) H_3''''$$

and  $H_1'$  through  $H_4''$  are transformation matrices such that



$$H'_1 = H'_3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2}\sqrt{\frac{1}{3}} & -\frac{1}{2}\sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2}\sqrt{\frac{2}{3}} & -\frac{1}{2}\sqrt{\frac{1}{3}} \end{bmatrix}$$

$$H'_2 = -H'_4 = \begin{bmatrix} -\frac{1}{2}\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{2}\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{\frac{1}{3}} & -\sqrt{\frac{2}{3}} & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

$$H''_1 = H''_3 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\sqrt{\frac{1}{3}} & \frac{1}{2}\sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{3}} \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2}\sqrt{\frac{2}{3}} & \frac{1}{2}\sqrt{\frac{1}{3}} \end{bmatrix}$$

$$H_2' = -H_4' = \begin{bmatrix} \frac{1}{2}\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{2}\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & -\frac{1}{2} & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$H_1'' = H_3'' = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\sqrt{\frac{1}{3}} & \frac{1}{2}\sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2}\sqrt{\frac{1}{3}} & \frac{1}{2}\sqrt{\frac{2}{3}} \end{bmatrix}$$

$$H_2'' = -H_4'' = \begin{bmatrix} \frac{1}{2}\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{2}\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$H_1'' = H_4'' = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\sqrt{\frac{1}{3}} & -\frac{1}{2}\sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2}\sqrt{\frac{2}{3}} & -\frac{1}{2}\sqrt{\frac{1}{3}} \end{bmatrix}$$

$$H_2'' = H_4'' = \begin{bmatrix} \frac{1}{2}\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{2}\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

and all other variables are defined in the previous section on the transfer matrix  $V_2$ .

LENGTH OF EACH BAR =  $l$

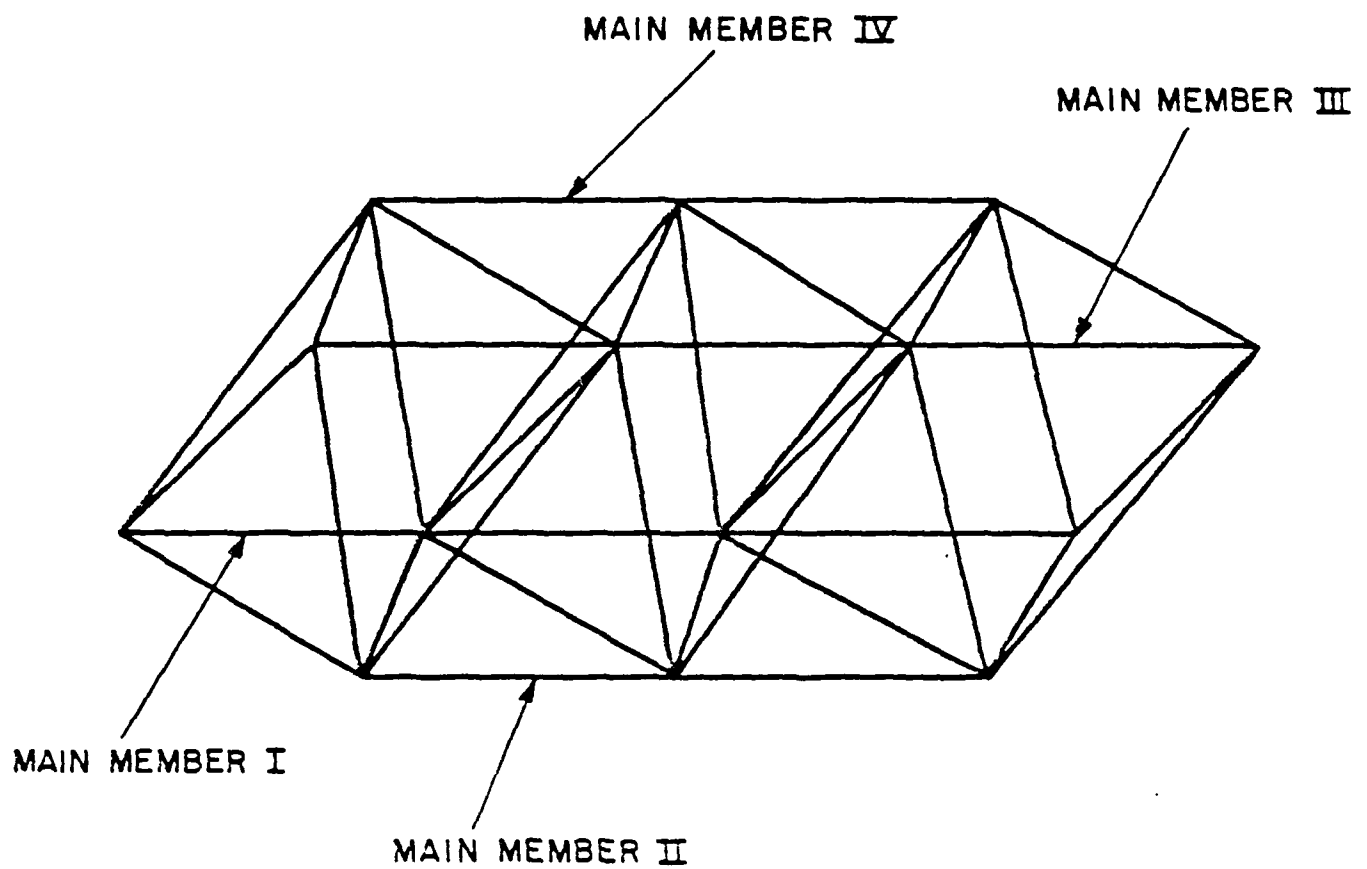


Fig. I 1 A tetrahedral truss.

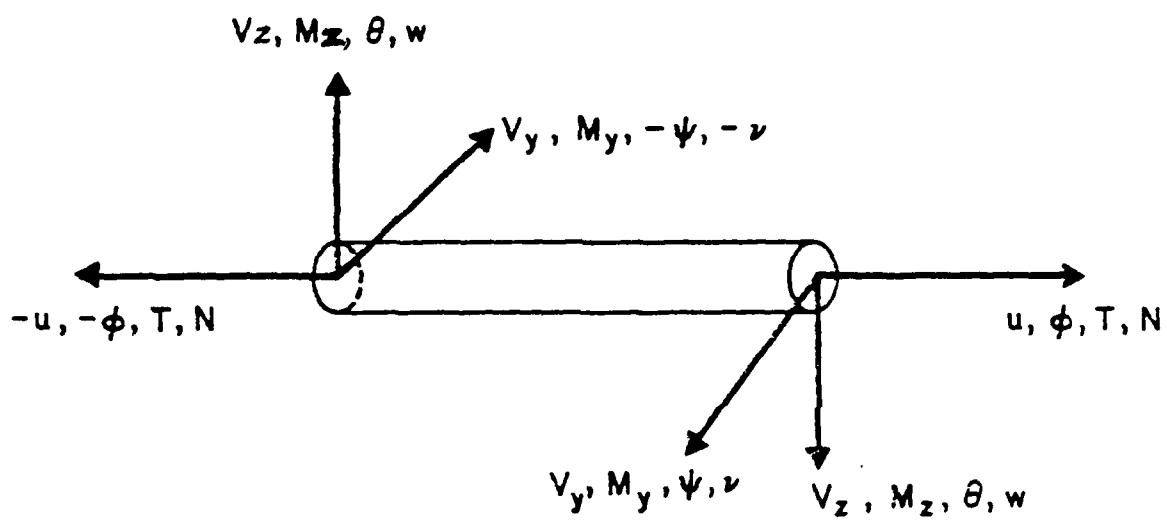


Fig. I2 Sign convention for the forces and displacements in a connecting bar.

LENGTH OF EACH BAR =  $l$

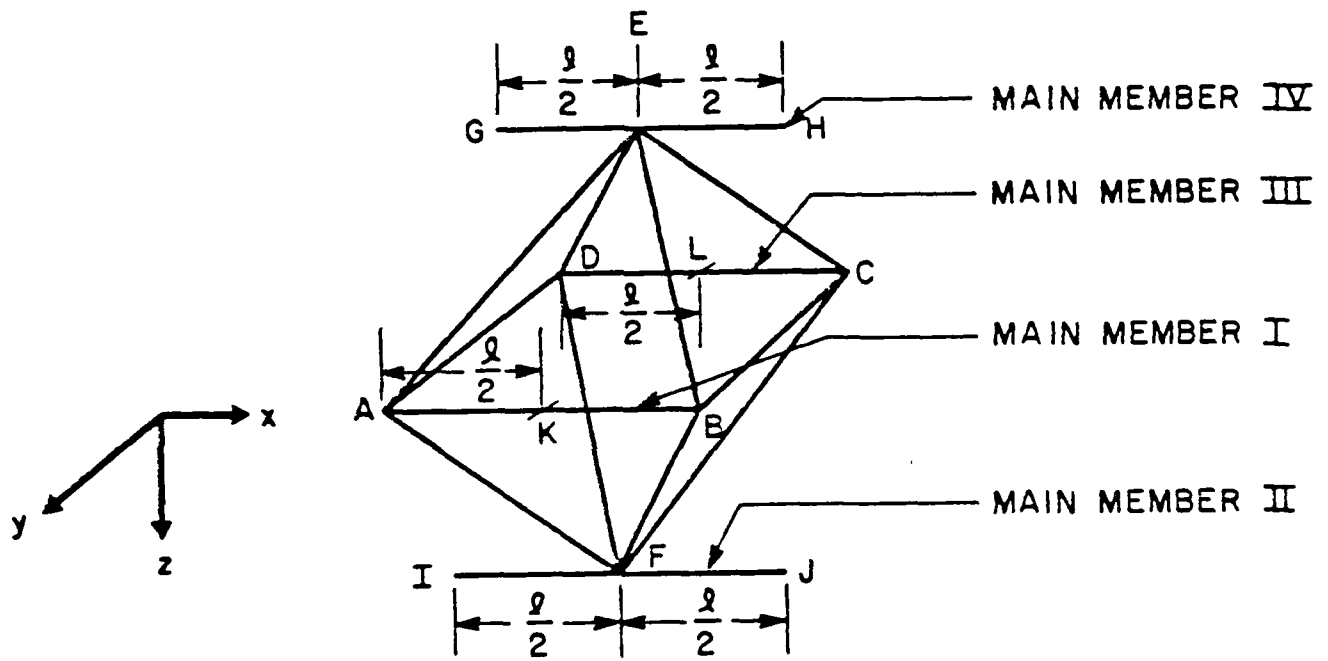


Fig. I3 A periodic unit of the tetrahedral truss.

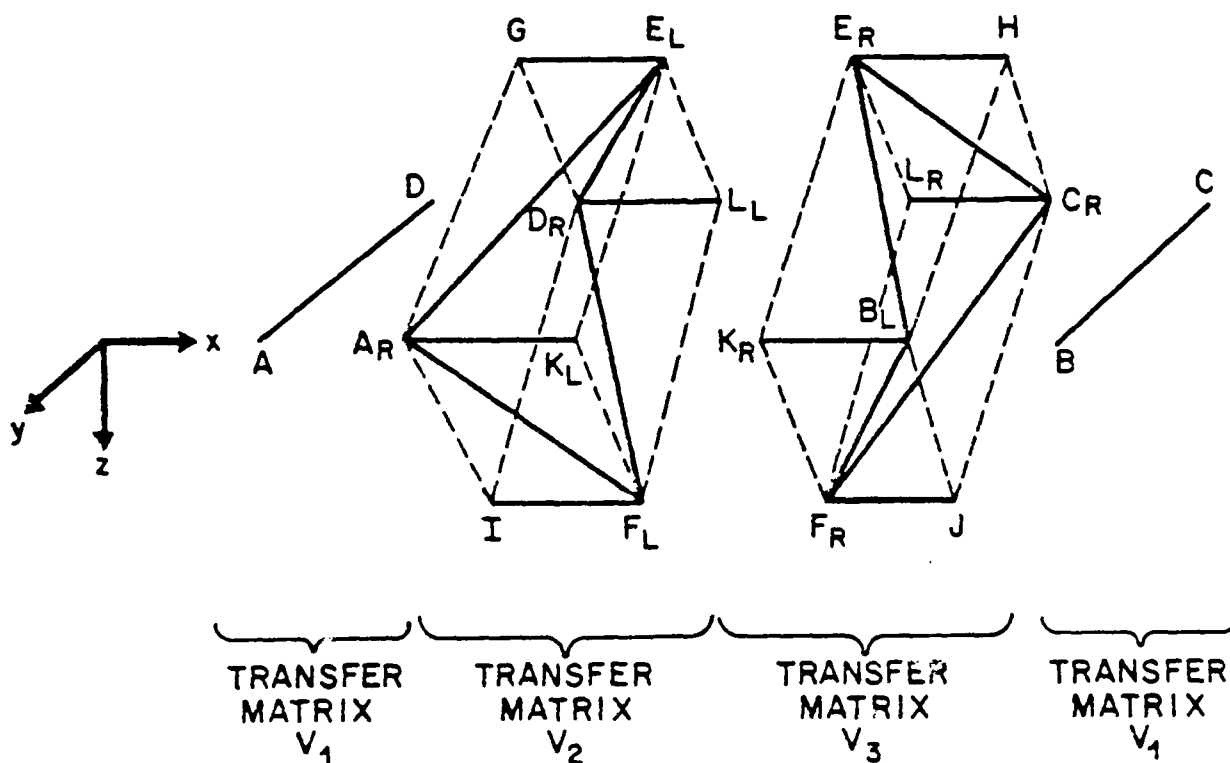


Fig. I4 Sectioning of a periodic unit into constituent parts comprising the transfer matrices.

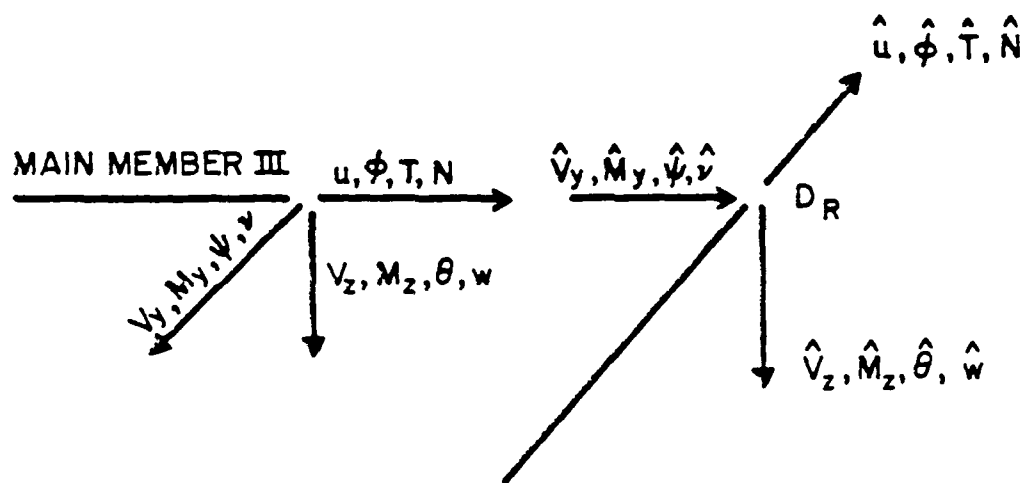


Fig. I5 Directional relations between global and local state vectors at location D.



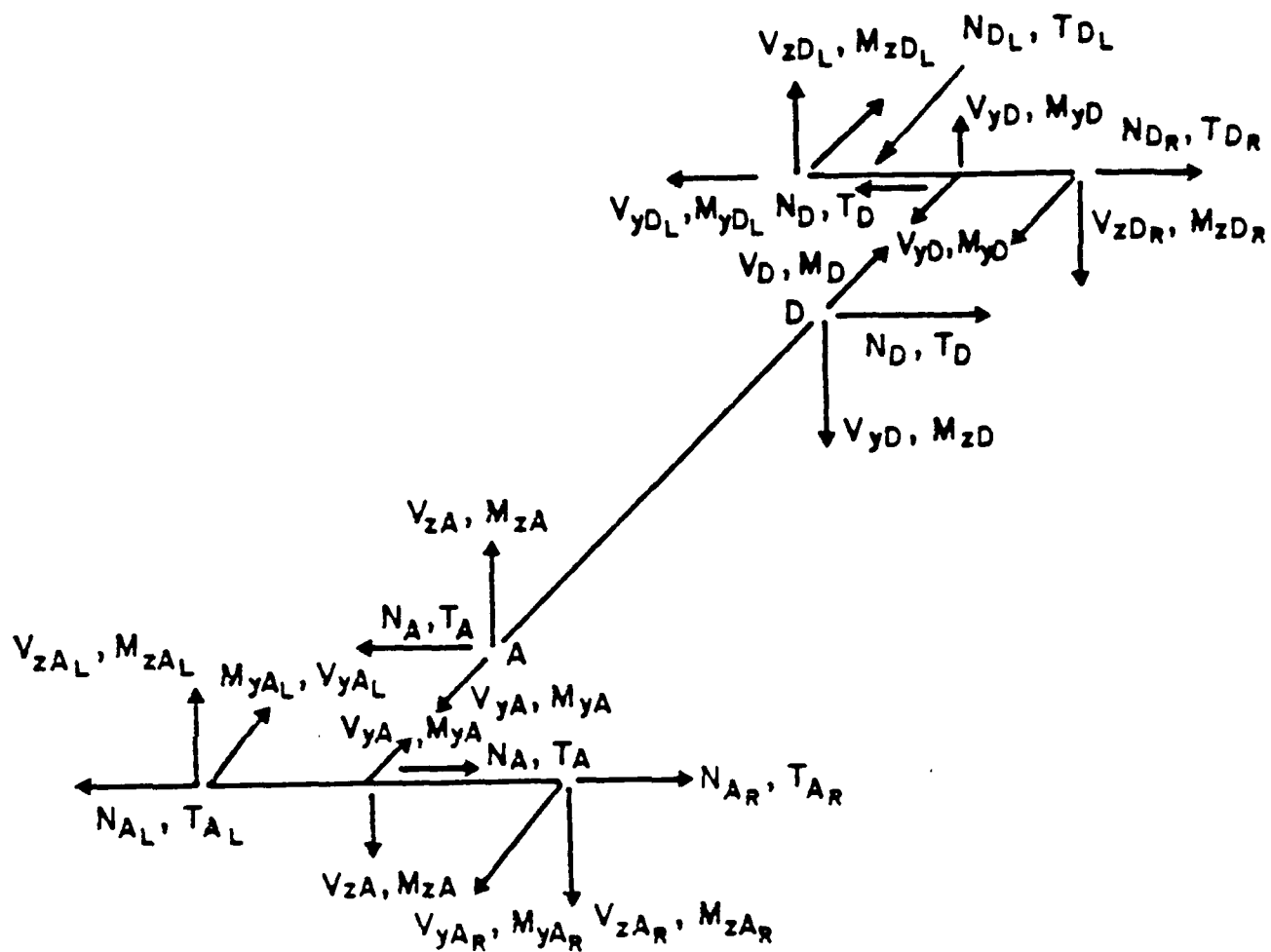


Fig. I6 Forces at locations A and D.

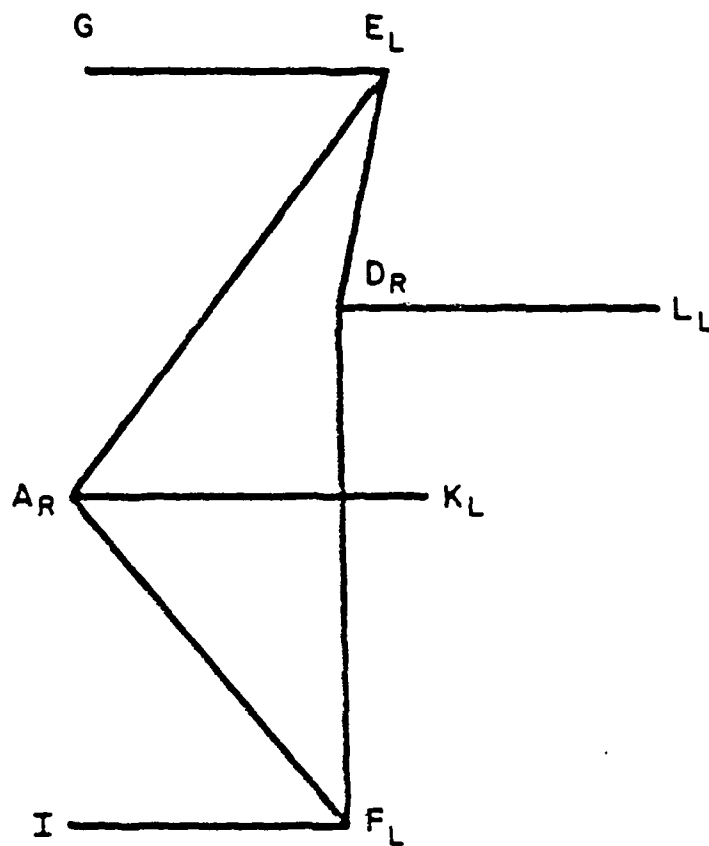


Fig. I7 Tetrahedral truss section for matrix  $V_2$ .

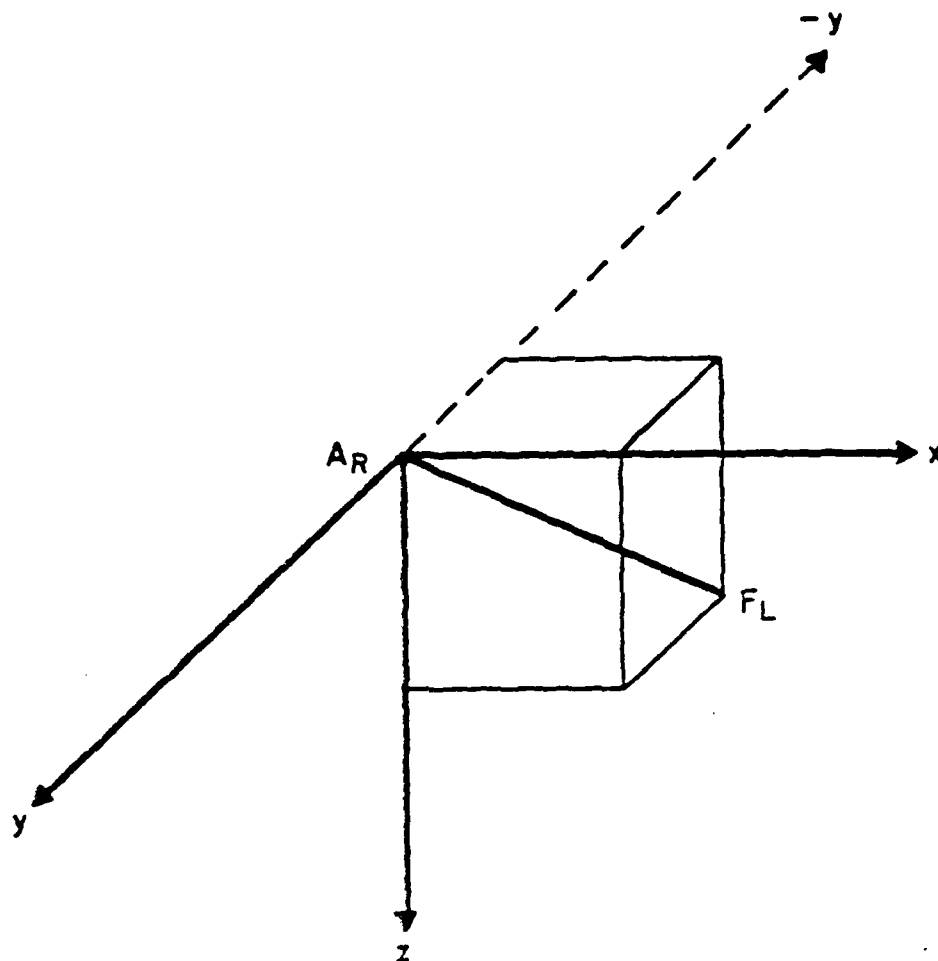


Fig. I8 Orientation of member  $A_R F_L$  in the global  $xyz$  coordinates.

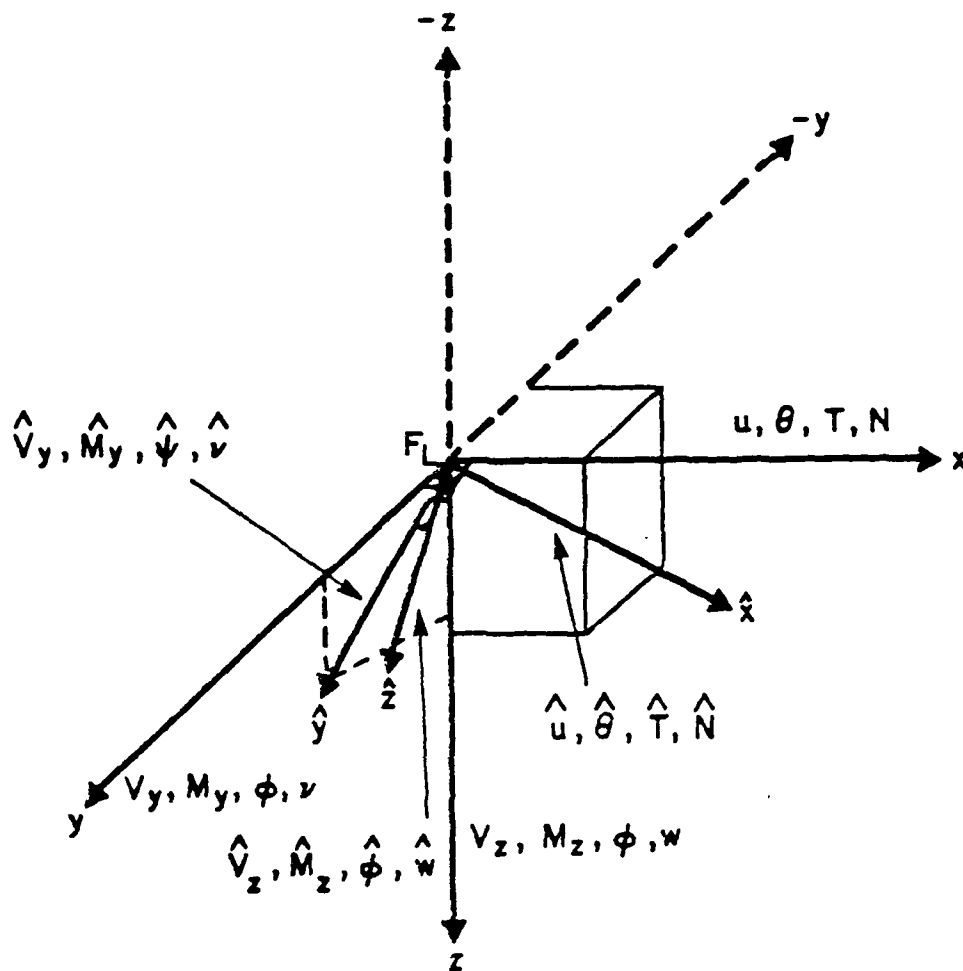


Fig. I9 Relationships between state vectors in the global coordinate and the state vectors in the local coordinates at location  $F_L$ .

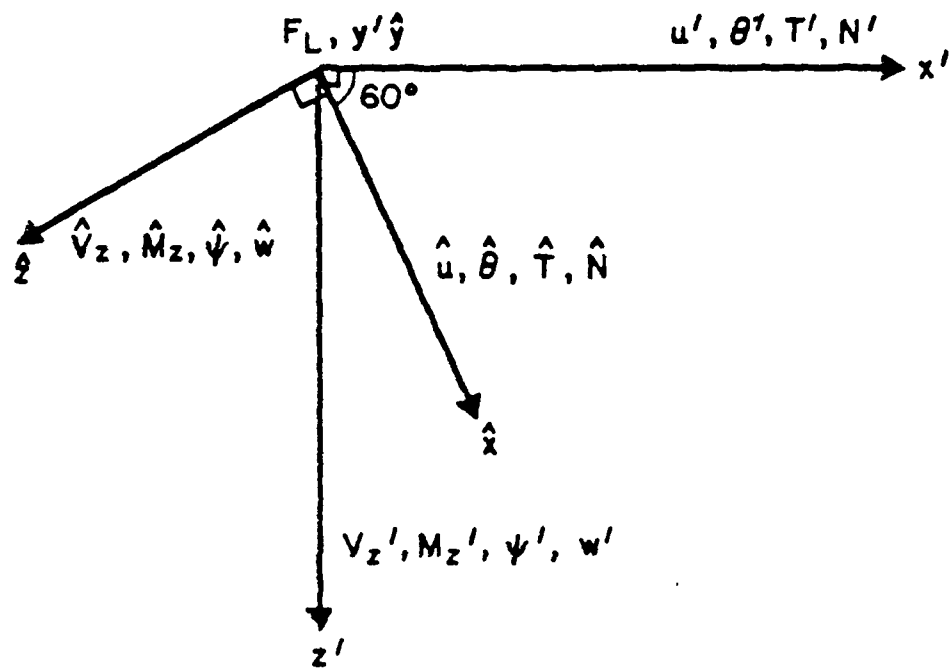
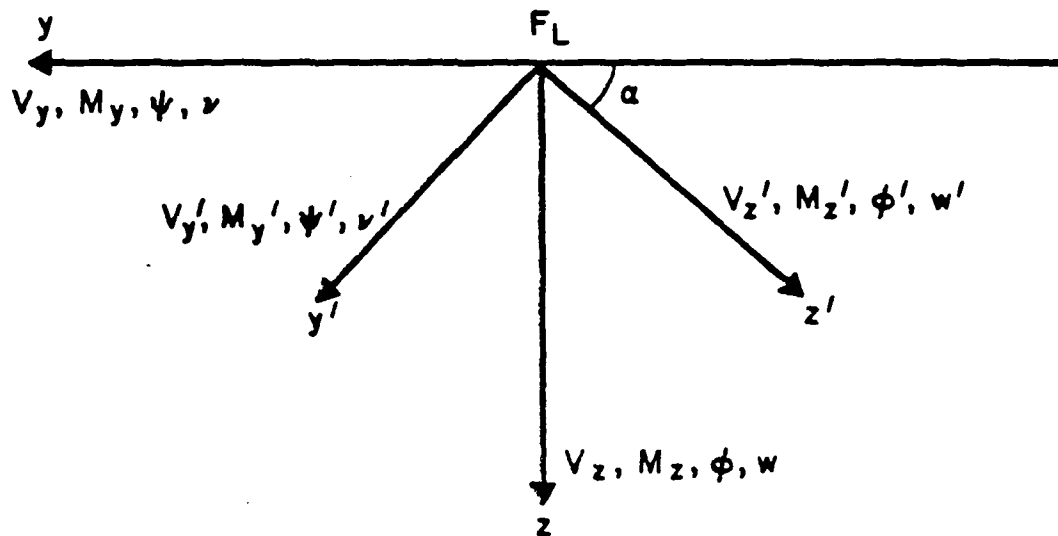


Fig. I10 Directional relations between the local and the rotated global state vectors at location  $F_L$ .



$$\sin \alpha = \frac{\frac{1}{\sqrt{2}}}{\frac{\sqrt{3}}{2}} = \sqrt{\frac{2}{3}}$$

$$\cos \alpha = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

Fig. I11 Directional relations between the rotated and unrotated global state vectors at location  $F_L$ .

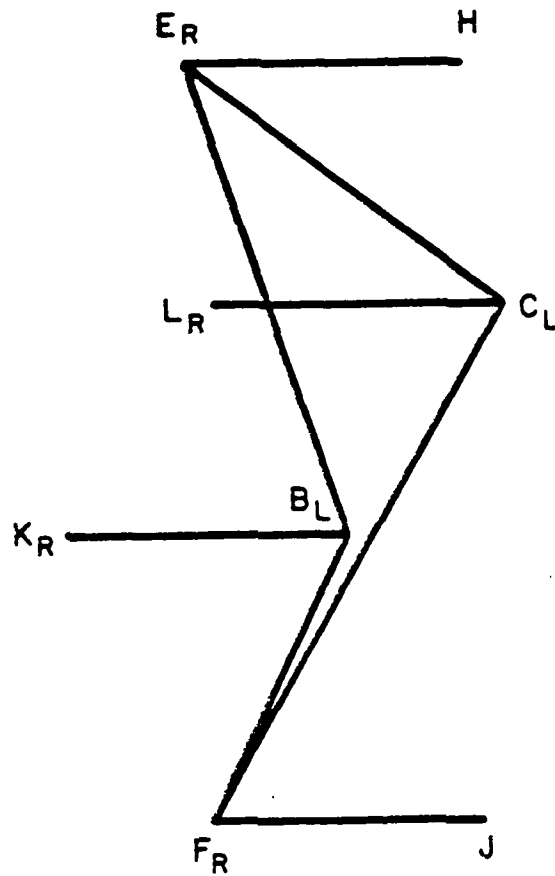


Fig. I12 Tetrahedral truss section for matrix  $V_3$ .

*APPENDIX J*

*LIST OF COMPUTER PROGRAM*

This section gives a Basic language computer program, named PROG1, BAS, which calculate the frequency response functions for an elastic rod. Fig. J1 shows a schematic of the rod and the symbols for various locations used. PROG1.BAS calculates the frequency response functions, according to discrete frequency steps, for locations  $A, B, C, D, E_L, E_R, F$  and  $G$ , where  $E_L$  and  $E_R$  are points just left of  $E$  and just right of  $E$ , respectively. The values of the frequency response functions are stored, according to frequency, onto files, one file for each location. The letter before the decimal point in each file name denotes the response location and the word used after the decimal point denotes the type of response which is being stored. If a file name ends with the word one, the response of the longitudinal displacement is stored whereas, if a file name ends with the word two, the response of the axial force is stored.



# PROG1.BAS

```
on error goto 84
defdbl t,e,w,m,a,b,l
defint i,j
option base 1
dim t(2,2)
read e,m,l,a
data 10.8e6,2.6e-4,10,9.75e-2
input "Enter the starting point      : ",k1 : print
input "Enter the terminating point : ",k3 : print
10 for k2=k1 to k3 step 10
  w=k2*100
  for i=1 to 2
    for j=1 to 2
      t(i,j)=0
    next j
  next i
  l=l*2
  gosub 1000
  b1=t(2,2)
  l=l/2*6
  gosub 1000
  a1=t(2,2)
  open "a", 1, "freqa.one"
  write #1, w,-b1/a1
  close 1
  l=l/6
  gosub 1000
  open "a", 1, "freqb.one"
  write #1, w,-b1/a1*t(1,2)
  close 1
  open "a", 1, "freqb.two"
  write #1, w,-b1/a1*t(2,2)
  close 1
  l=l*2
  gosub 1000
  open "a", 1, "freqc.one"
  write #1, w,-b1/a1*t(1,2)
  close 1
  open "a", 1, "freqc.two"
  write #1, w,-b1/a1*t(2,2)
  close 1
  l=l/2*3
  gosub 1000
  open "a", 1, "freqd.one"
  write #1, w,-b1/a1*t(1,2)
  close 1
  open "a", 1, "freqd.two"
  write #1, w,-b1/a1*t(2,2)
  close 1
  l=l/3*4
  gosub 1000
```

```
open "a", 1, "freqel.one"
write #1, w, -b1/a1*t(1,2)
close 1
open "a", 1, "freqel.two"
write #1, w, -b1/a1*t(2,2)
close 1
b2=-t(1,2)
l=1/4*6
gosub 1000
a2=t(1,1)
open "a", 1, "freqg.one"
write #1, w, b2/a2
close 1
l=1/6
gosub 1000
open "a", 1, "freqf.one"
write #1, w, b2/a2*t(1,1)
close 1
open "a", 1, "freqf.two"
write #1, w, -b2/a2*t(2,1)
close 1
l=1*2
gosub 1000
open "a", 1, "frequer.one"
write #1, w, b2/a2*t(1,1)
close 1
open "a", 1, "frequer.two"
write #1, w, -b2/a2*t(2,1)
close 1
l=1/2
next k2
goto 86
84 k1=k2+10
resume 10
86 end
1000 th=l*w*sqr(m*a/e)
t(1,1)=cos(th)
t(1,2)=l*sin(th)/e/a/th
t(2,1)=-m*l*w*w*sin(th)/th
t(2,2)=t(1,1)
return
```

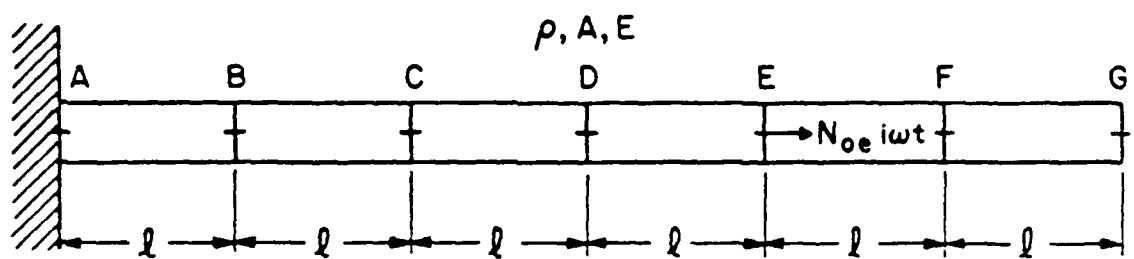


Fig. J1 An elastic rod loaded with a sinusoidal force at E.

## APPENDIX K

### SOME PROPERTIES OF TRANSFER MATRICES

#### CROSS-SYMMETRY

If a segment of an element is symmetric about a plane at its midlength as shown schematically in Fig. K1, then it is always possible to obtain a cross-symmetric transfer matrix, which represents the element, by a suitable ordering and sign convention of the components of the state vector [7]. The transfer matrix is cross-symmetric in that its elements are symmetric about its cross-diagonal. Thus, if the elements in a cross-symmetric transfer matrix  $T$  are  $t_{ij}$ , where  $i=1,2,\dots,n$  for the rows and  $j=1,2,\dots,n$  for the columns and  $n$  is the dimension of the transfer matrix (transfer matrices are square matrices), then

$$t_{ij} = t_{n+1-i, n+1-j}. \quad (K1)$$

For example, the transfer matrix for an elastic rod can be observed to be cross-symmetric using the state vector as defined. Similarly, it can be shown that the transfer matrix for a Timoshenko beam is cross-symmetric with a suitable ordering of the components of the state vector.

#### INVERSION

For a given element, there exists two possible transfer matrices to relate either (1) the state vector at the right end to the state vector at the left end, or (2) the state vector at the left end to the state vector at the right end. Let  $T$  be a transfer matrix which relates the state vector at the right end to the state vector at the left end of an element. Thus,

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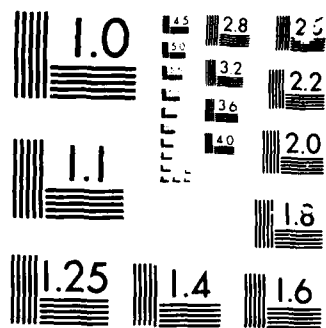
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$$z_R = T z_L \quad (K2)$$

where  $z_R$  and  $z_L$  denote the state vector at the right end and the state vector at the left end of the element, respectively. Eqn. (K2) can be rearranged to yield

$$z_L = T^{-1} z_R \quad (K3)$$

Eqn. (K3) shows that the state vector at the left end is related to the state vector at the right end by the inverse of the transfer matrix  $T$ .

For example, the transfer matrix relating the state vector at the left end to the right end of an elastic rod is the inverse of the transfer matrix relating the state vector at the right end to the left end of the same rod.

It may also be shown that transfer matrices are non-singular and that the inverse of a transfer matrix is equal to its adjoint [7].

#### VALUE OF DETERMINANT

It can be shown that the determinant of any transfer matrix is equal to unity [7].

For example, the determinant of the transfer matrix for the longitudinal vibration of an elastic rod (or the inverse of the transfer matrix) can be shown to be unity algebraically.

Because digital computations are necessary in applying the transfer matrix method, the numerical evaluation of the determinant of the transfer matrix provides an opportunity to determine the accuracy of computer-aided numerical results. Table K1 shows the listing of a Basic language computer program named DET1.BAS, which calculates the determinant of the transfer matrix for the longitudinal vibration of an elastic rod at various frequencies. Table K2 shows the listing of a Basic language computer program named DET2.BAS, which calculates the determinant of the transfer matrix for the flexural vibration of a Timoshenko beam at various frequencies. The

programs are evaluated using double precision on all variables and calculations; that is, all variables and calculations are carried out to 16 significant figures. Table K3 shows the computer values of the determinant for the rod and the Timoshenko beam at various frequencies. From Table K3, it is observed that the determinant of the rod is always calculated correctly as unity. However, it is observed that the determinant of the Timoshenko beam is calculated to be unity only at low frequencies such as less than approximately 6000 rad/sec (approximately 1000 Hz).

### EIGENVALUES

It can be shown that the eigenvalues (that is, the natural frequencies) corresponding to vibrations of a structure evaluated using the transfer matrices relating (1) the state vector at the right end to the state vector at the left end, and (2) the state vector at the left end to the state vector at the right end, are identical [7].

For example, the natural frequencies of longitudinal vibration in an elastic rod shown in Fig. K2 can be evaluated using the transfer matrices for the rod. As shown in Fig. K2, the rod is clamped at the left end and is free at the right end. The state vector of the rod at the right end is related to the state vector at the left end by a transfer matrix as

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_R = \begin{bmatrix} \cos \theta & \frac{\ell}{EA} \frac{\sin \theta}{\theta} \\ -\mu \ell \omega^2 \frac{\sin \theta}{\theta} & \cos \theta \end{bmatrix} \begin{Bmatrix} u \\ N \end{Bmatrix}_L \quad (K4)$$

where  $u$  is the longitudinal displacement  $N$  is the axial force and the subscripts  $R$  and  $L$  denote the right end and the left end of the rod, respectively. Imposing the boundary conditions of



$$u_L = 0, \text{ at } x = 0 \quad (K5)$$

$$\text{and } N_R = 0, \text{ at } x = \ell \quad (K6)$$

into eqn. (K4) gives

$$\begin{Bmatrix} u \\ 0 \end{Bmatrix}_R = \begin{bmatrix} \cos \theta & \frac{\ell \sin \theta}{EA} \\ -\mu \ell \omega^2 \frac{\sin \theta}{\theta} & \cos \theta \end{bmatrix} \begin{Bmatrix} 0 \\ N \end{Bmatrix}_L \quad (K7)$$

Eqn. (K7) can be rewritten as the following two equations:

$$u_R = \frac{\ell \sin \theta}{EA} N_L \quad (K8)$$

$$\text{and } 0 = \cos \theta N_L \quad (K9)$$

From eqn. (K9) nontrivial solutions require that

$$\cos \theta = 0 \quad (K10)$$

$$\text{or } \theta = \frac{(2n-1)\pi}{2} \text{ for } n=1,2,\dots \quad (K11)$$

$$\text{since } \theta = \ell \omega \sqrt{\frac{\rho}{E}}, \quad (K12)$$

substitution of eqn. (K12) into eqn. (K11) gives

$$\ell \omega \sqrt{\frac{\rho}{E}} = \frac{(2n-1)\pi}{2} \text{ for } n=1,2 \quad (\text{K13})$$

From eqn. (K13), the natural frequencies for longitudinal vibration of an elastic rod is

$$\omega = \frac{(2n-1)\pi}{2} \sqrt{\frac{E}{\rho}} \text{ for } n=1,2,\dots \quad (\text{K14})$$

Similarly, the state vector of the rod at the left end is related to the state vector at the right end by a transfer matrix as

$$\begin{Bmatrix} u \\ N \end{Bmatrix}_L = \begin{bmatrix} \cos \theta & -\frac{\ell}{EA} \frac{\sin \theta}{\theta} \\ \mu \ell \omega^2 \frac{\sin \theta}{\theta} & \cos \theta \end{bmatrix} \begin{Bmatrix} u \\ N \end{Bmatrix}_R \quad (\text{K15})$$

Imposing the boundary conditions in eqns. (K5) and (K6) gives

$$\begin{Bmatrix} 0 \\ N \end{Bmatrix}_L = \begin{bmatrix} \cos \theta & -\frac{\ell}{EA} \frac{\sin \theta}{\theta} \\ \mu \ell \omega^2 \frac{\sin \theta}{\theta} & \cos \theta \end{bmatrix} \begin{Bmatrix} u \\ 0 \end{Bmatrix}_R \quad (\text{K16})$$

Eqn. (K16) can be rewritten as the following two equations

$$0 = \cos \theta u_R \quad (\text{K17})$$

$$\text{and } N_L = \mu \ell \omega^2 \frac{\sin \theta}{\theta} u_R \quad (\text{K18})$$

From eqn. (K17), nontrivial solutions require that

$$\cos \theta = 0 \quad (K19)$$

Since eqns. (K10) and (K19) are identical, the same natural frequencies as given in eqn. (K14) will result. Thus, it is demonstrated that the same eigenvalues (or natural frequencies) are obtained using the transfer matrices relating either (1) the state vector at the right end to the state vector at the left end, or (2) the state vector at the left end to the state vector at the right end.

Table K1. Computer listing of program DET1.BAS

```
defint i,j
defdbl a
option base 1
dim a(2,2)
read e,ar,ro,l
data 10e7,1.84e-1,3e-4,9.75
input "The frequency of vibration is : ";w : print
lprint "The frequency of vibration is : ";w : lprint
for i=1 to 2
  for j=1 to 2
    a(i,j)=0
  next j
next i
gosub 1000
det=a(1,1)*a(2,2)-a(1,2)*a(2,1)
lprint "The determinant of transfer matrix is ";det : lprint
end
1000 th=1*w*sqr(ro/e)
a(1,1)=cos(th)
a(2,2)=a(1,1)
a(1,2)=1/e/ar/th*sin(th)
a(2,1)=-e*ar*th/l*sin(th)
return
```

Table K2. Computer listing of program DET2.BAS

```

defint i,j,m,n,k
defdbl a,b,c,e,g,r
option base 1
dim a(12,12),b(12,12)
def fncsh(f)=(exp(f)+exp(-f))/2
def fnsnh(h)=(exp(h)-exp(-h))/2
read e,g,ar,bi,ro,r,l,ga
data 10e7,3.84e7,1.84e-1,2.85e-3,3e-4,1.24e-1,9.75,5.92e6
input "Enter the frequency of vibration : ";w
lprint "The frequency of vibration is = ";w
gosub 2000
for i=1 to 12
for j=1 to 12
a(i,j)=0
b(i,j)=0
next j
next i
gosub 3000
for i=1 to 4
for j=1 to 4
b(i,j)=a(i+4,j+4)
next j
next i
lprint "The beam transfer matrix is :--- "
for i=1 to 4
lprint b(i,1);b(i,2);b(i,3);b(i,4)
next i
d1=(b(1,1)*b(2,2)-b(1,2)*b(2,1))*(b(4,4)*b(3,3)-b(3,4)*b(4,3))
d2=(b(1,1)*b(2,3)-b(2,1)*b(1,3))*(b(3,2)*b(4,4)-b(4,2)*b(3,4))
d3=(b(1,1)*b(2,4)-b(2,1)*b(1,4))*(b(3,2)*b(4,3)-b(4,2)*b(3,3))
d4=(b(1,2)*b(2,3)-b(2,2)*b(1,3))*(b(3,1)*b(4,4)-b(4,1)*b(3,4))
d5=(b(1,2)*b(2,4)-b(2,2)*b(1,4))*(b(3,1)*b(4,3)-b(4,1)*b(3,3))
d6=(b(1,3)*b(2,4)-b(2,3)*b(1,4))*(b(3,1)*b(4,2)-b(4,1)*b(3,2))
det=d1-d2+d3+d4-d5+d6
lprint "The determinant of the beam transfer matrix is = ";det
end
2000 su=ro/ar
a1=(1^2)/e/bi
b4=su*(w^2)*(1^4)/e/bi
s=su*(w^2)*(1^2)/ga
t=su*(r^2)*(w^2)*(1^2)/e/bi
th=1*w*sqr(ro/e)
l1=sqr(sqr(b4+(((s-t)^2)/4))-(s+t)/2)
l2=sqr(sqr(b4+(((s-t)^2)/4))+(s+t)/2)
l0=1/((l1^2)+(l2^2))
c0=l0*((l2^2)*fncsh(l1)-(l1^2)*cos(l2))
c1=l0*((l2^2)/l1*fnsnh(l1)+(l1^2)/l2*sin(l2))
c2=l0*(fncsh(l1)-cos(l2))
c3=l0*(1/l1*fnsnh(l1)-1/l2*sin(l2))
return

```

Table K2. Cont. Computer listing of program DET2.BAS

```
rem subroutine to form beam transfer matrix
3000 for i=1 to 12
  for j=1 to 12
    a(i,j)=0
  next j
next i
a(1,1)=cos(th)
a(4,4)=a(1,1)
a(9,9)=a(1,1)
a(12,12)=a(1,1)
a(1,12)=1*sin(th)/(e*ar*th)
a(4,9)=a(1,12)
a(2,2)=c0-s*c2
a(5,5)=a(2,2)
a(2,3)=1*(c1-(s+t)*c3)
a(5,6)=a(2,3)
a(2,10)=a1*c2
a(3,11)=a(2,10)
a(5,7)=a(2,10)
a(6,8)=a(2,10)
a(2,11)=a1*1/b4*(-s*c1+(b4+s^2)*c3)
a(5,8)=a(2,11)
a(3,2)=b4/1*c3
a(6,5)=a(3,2)
a(3,3)=c0-t*c2
a(6,6)=a(3,3)
a(3,10)=a1/1*(c1-t*c3)
a(6,7)=a(3,10)
a(7,5)=b4*c2/a1
a(8,6)=a(7,5)
a(10,2)=a(7,5)
a(11,3)=a(7,5)
a(7,6)=1/a1*(-t*c1+(b4+t^2)*c3)
a(10,3)=a(7,6)
a(7,7)=c0-t*c2
a(10,10)=a(7,7)
a(7,8)=1*(c1-(s+t)*c3)
a(10,11)=a(7,8)
a(8,5)=b4*(c1-s*c3)/a1/1
a(11,2)=a(8,5)
a(8,7)=b4*c3/1
a(11,10)=a(8,7)
a(8,8)=c0-s*c2
a(11,11)=a(8,8)
a(9,4)=-su*1*(w^2)*sin(th)/th
a(12,1)=a(9,4)
return
```

Table K3. Computed Values of the Determinant of Transfer Matrices for Uniform Rod and Timoshenko Beam for Various Frequencies of Vibration\*

Frequency (rad/sec)	Computed Values of the Determinant of Transfer Matrix for	
	Longitudinal Vibration of Uniform Rod	Flexural Vibration of Uniform Timoshenko Beam
10	1	1
100	1	1
1,000	1	0.99999
10,000	1	-0.0625
100,000	1	$1.80144 \times 10^{16}$
1,000,000	1	**

\* For the rod, elastic modulus is  $7.46 \times 10^{10}$  Pa ( $10.8 \times 10^6$  psi), cross-sectional area is  $6.04 \times 10^{-5} \text{ m}^2$  ( $9.375 \times 10^{-2} \text{ in}^2$ ), mass density is  $7.2 \text{ kg/m}^3$  ( $2.6 \times 10^{-4} \text{ lbm/in}^3$ ), and length is  $2.43 \times 10^{-1} \text{ m}$  (9.75 in).

For the Timoshenko beam, elastic modulus is  $7.46 \times 10^{10}$  Pa ( $10.8 \times 10^6$  psi), shear modulus is  $2.75 \times 10^{10}$  Pa ( $4.0 \times 10^6$  psi), cross-sectional area is  $6.04 \times 10^{-5} \text{ m}^2$  ( $9.375 \times 10^{-2} \text{ in}^2$ ), second moment of area inertia is  $4.55 \times 10^{-10} \text{ m}^4$  ( $1.098 \times 10^{-3} \text{ in}^4$ ), mass density is  $7.2 \text{ kg/m}^3$  ( $2.6 \times 10^{-4} \text{ lbm/in}^3$ ), radius of gyration is  $2.71 \times 10^{-3} \text{ m}$  ( $1.0838 \times 10^{-1} \text{ in}$ ), and length is  $2.43 \times 10^{-1} \text{ m}$  (9.75 in).

\*\* Value exceeded the capacity of the computer

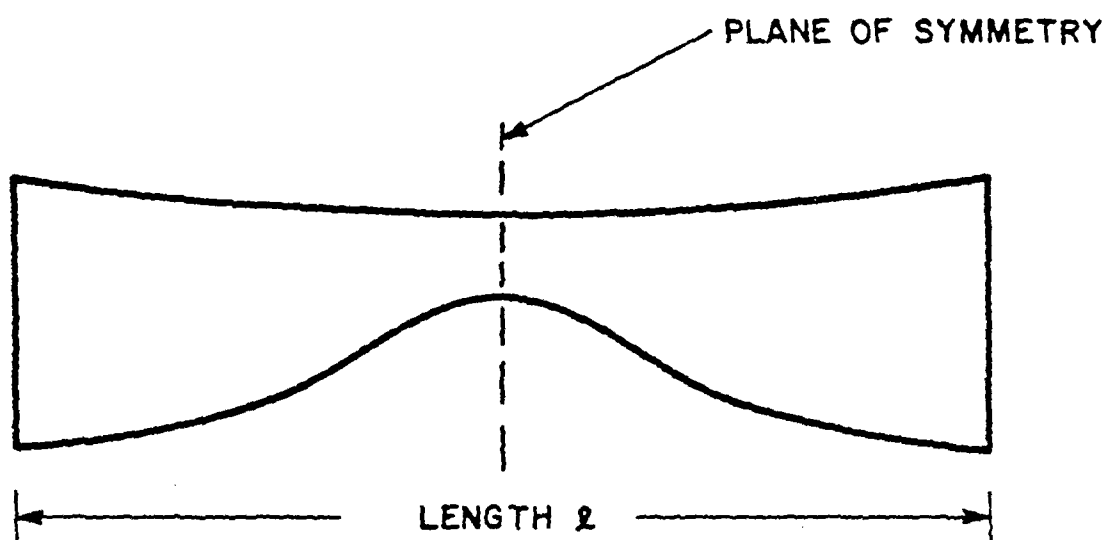


Fig. K1 Segment of an element represented by a cross-symmetric transfer matrix.



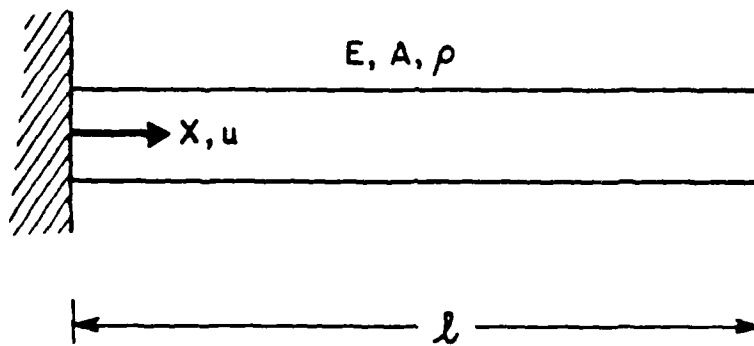


Fig. K2 An elastic rod constrained at one end and free at the other end.

APPENDIX L

NON-DIMENSIONALIZED FORMS FOR TRANSFER MATRICES OF A  
3-BAY PLANAR LATTICE STRUCTURE

The non-dimensionalized forms for the two transfer matrices  $X_1$  and  $X_2$  for the analysis of wave propagation and vibration in a 3-bay planar structure are derived in this appendix.

From eqn. (39) and eqn. (40), the two transfer matrices  $X_1$  and  $X_2$ , corresponding to a state vector  $Z$  such that

$$Z = \begin{Bmatrix} d_I \\ d_{II} \\ p_{II} \\ p_I \end{Bmatrix} \quad (L1)$$

where

$$d = \begin{Bmatrix} u \\ -w \\ \psi \end{Bmatrix}$$

$$\text{and where } p = \begin{Bmatrix} M \\ V \\ N \end{Bmatrix}$$

and where the subscripts denote the member numbers, are given by

$$X_1 = \begin{bmatrix} C_1 & 0 & 0 & C_2 \\ 0 & C_1 & C_2 & 0 \\ 0 & C_3 & C_4 & 0 \\ C_3 & 0 & 0 & C_4 \end{bmatrix} \quad (L2)$$

$$\text{and } X_2 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ G_4 C_2^{-1} G_1 & -G_4 C_2^{-1} C_1 G_3 & I & 0 \\ G_2 C_4 C_2^{-1} G_1 & G_2 (C_3 - C_4 C_2^{-1} C_1) G_3 & 0 & I \end{bmatrix} \quad (L3)$$

I is the identity matrix, 0 is the null matrix,  $C_1, C_2, C_3$  and  $C_4$  are  $3 \times 3$  submatrices and  $G_1, G_2, G_3$  and  $G_4$  are transformation matrices such that

$$C_1 = \begin{bmatrix} \cos \theta & 0 & 0 \\ 0 & c_0 - \alpha_2 & \ell [c_1 - (\sigma + \tau) c_3] \\ 0 & \frac{\beta^4}{\ell} c_3 & c_0 - \tau c_2 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0 & 0 & \frac{\ell}{EA} \frac{\sin \theta}{\theta} \\ \frac{\ell^2}{EJ} c_2 & \frac{\ell^3}{\beta^4 EJ} [-\sigma c_1 + (\beta^4 + \sigma^2) c_3] & 0 \\ \frac{\ell}{EJ} (c_1 - \tau c_3) & \frac{\ell^2}{EJ} c_2 & 0 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} 0 & \frac{\beta^4 EJ}{\ell^2} c_2 & \frac{EJ}{\ell} [-\tau c_1 + (\beta^4 + \tau^2) c_3] \\ 0 & \frac{\beta^4 EJ}{\ell^3} [c_1 - \sigma c_3] & \frac{\beta^4 EJ}{\ell^2} c_2 \\ -\mu \ell \omega^2 \frac{\sin \theta}{\theta} & 0 & 0 \end{bmatrix}$$

$$C_4 = \begin{bmatrix} c_0 - \tau c_2 & \ell [c_1 - (\sigma + \tau) c_3] & 0 \\ \frac{\beta^4}{\ell} c_3 & c_0 - \sigma c_2 & 0 \\ 0 & 0 & \cos \theta \end{bmatrix}$$

$$G_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$G_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G_4 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

where

$$\theta = \ell \omega \sqrt{\frac{\rho}{E}}$$

$$\Lambda = \frac{1}{\lambda_1^2 + \lambda_2^2}$$

$$\lambda_2^1 = \sqrt{\sqrt{\beta^4 + \frac{1}{4}(\sigma - \tau)^2} \mp \frac{1}{2}(\sigma + \tau)}$$

$$c_0 = \Lambda(\lambda_2^2 \cosh \lambda_1 + \lambda_1^2 \cos \lambda_2)$$

$$c_1 = \Lambda\left(\frac{\lambda_2^2}{\lambda_1} \sinh \lambda_1 + \frac{\lambda_1^2}{\lambda_2} \sin \lambda_2\right)$$

$$c_2 = \Lambda(\cosh \lambda_1 - \cos \lambda_2)$$

$$c_3 = \Lambda\left(\frac{\sinh \lambda_1}{\lambda_1} - \frac{\sin \lambda_2}{\lambda_2}\right)$$

where

$$\sigma = \mu \omega^2 \frac{\ell^2}{GA_s}$$

$$\tau = \frac{\mu \ell^2 i^2 \omega^2}{EJ}$$

$$\beta^4 = \mu^2 \frac{\ell^4}{EJ}$$

Now assume that the connecting bars have identical square cross-sections of side  $a$  such that

$$J = \frac{a^4}{12} \quad (L4)$$

$$i = \frac{a}{\sqrt{12}} \quad (L5)$$

$$A = \frac{5}{6} a^2 \quad (L6)$$

From the elastic theory of isotropic materials,

$$G = \frac{E}{2(1+\nu)} \quad (L7)$$

where  $\nu$  is Poisson's ratio. Now let

$$\Omega^2 = \frac{\mu \omega^2 \ell a^2}{E} \quad (L8)$$

and

$$\ell a = \frac{\ell}{a} \quad (L9)$$

where  $\Omega$  and  $\ell a$  are both non-dimensionalized lumped parameters. Using Equation (L4) through Equation (L9), the variables  $\theta$ ,  $\sigma$ ,  $\tau$ , and  $\beta^4$  used in the transfer practices  $X_1$  and  $X_2$  can be rewritten as

$$\theta = \Omega \quad (L10)$$

$$\sigma = \frac{12}{5}(1+\nu)\Omega^2 \quad (L11)$$

$$\tau = \Omega^2 \quad (L12)$$

$$\beta^4 = 12\Omega^2 \ell a^2 \quad (L13)$$

Now replace the state vector  $Z$  given in Equation (L1) by a new non-dimensionalized state vector  $\bar{Z}$  such that

$$\bar{Z} = \begin{pmatrix} \bar{d}_I \\ \bar{d}_{II} \\ \bar{p}_{II} \\ \bar{p}_I \end{pmatrix} \quad (L14)$$

where

$$\bar{d} = \begin{pmatrix} \frac{a}{\ell} \\ -\frac{w}{\ell} \\ \psi \end{pmatrix}$$

and where

$$\bar{p} = \begin{pmatrix} \frac{M \ell}{EJ} \\ \frac{V \ell^2}{EJ} \\ \frac{N}{Ea^2} \end{pmatrix}$$

Equations (L2) and (L3) can now be rewritten as

$$X_1 = \begin{bmatrix} \bar{C}_1 & 0 & 0 & \bar{C}_2 \\ 0 & \bar{C}_1 & \bar{C}_2 & 0 \\ 0 & \bar{C}_3 & \bar{C}_4 & 0 \\ \bar{C}_3 & 0 & 0 & \bar{C}_4 \end{bmatrix} \quad (L15)$$

$$\text{and } X_2 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ G_4 \bar{C}_2^{-1} G_1 & -G_4 \bar{C}_2^{-1} \bar{C}_1 G_3 & I & 0 \\ G_2 \bar{C}_4 \bar{C}_2^{-1} G_1 & G_2 (\tau_3 - \bar{C}_4 \bar{C}_2^{-1} \bar{C}_1) G_3 & 0 & I \end{bmatrix} \quad (L16)$$

where

$$\bar{C}_1 = \begin{bmatrix} \cos \Omega & 0 & 0 \\ 0 & \cos \alpha_2 & [c_1 - (E + \tau) c_3] \\ 0 & \beta^4 c_3 & c_0 - \tau c_2 \end{bmatrix}$$

$$\bar{C}_2 = \begin{bmatrix} 0 & 0 & \frac{\sin \Omega}{\Omega} \\ c_2 & \frac{1}{\beta^4} [\alpha_1 + (\beta^4 + \sigma^2) c_3] & 0 \\ c_1 - \tau c_3 & c_2 & 0 \end{bmatrix}$$

$$\bar{C}_3 = \begin{bmatrix} 0 & \beta^4 c_2 & [-\tau c_1 + (\beta^4 + \tau^2) c_3] \\ 0 & \beta^4 (c_1 - \alpha c_3) & \beta^4 c_2 \\ \Omega \sin \Omega & 0 & 0 \end{bmatrix}$$

$$\bar{C}_4 = \begin{bmatrix} c_0 - \tau c_2 & [c_1 - (\sigma - \tau) c_3] & 0 \\ \beta^4 c_3 & c_0 - \alpha c_2 & 0 \\ 0 & 0 & \cos \Omega \end{bmatrix}$$

Notice that  $X_1$  and  $X_2$  in Equations (L15) and (L16) correspond to a non-dimensionalized state vector  $\bar{Z}$  and the elements in the two matrices are functions of non-dimensionalized lumped parameters  $\ell a$  and  $\Omega$ .

END

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